Quasi-systematic minimalism within socio-constructivist learning of mathematics.

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Abstract

In mathematics education research, the focus has generally been on a single factor or on a few partial factors at a time. Researchers have neglected the relationship between conceptual and procedural knowledge and how progressive technology alters this relation. The relationship between systematic planning of teaching and the reality of often chaotic learning situations has not been properly explored. There is no pragmatic theory of collaborative socio-constructivist student-centered learning, especially with regard to the use of technology. The studies in this article were conducted in order to consider the effects of a minimalist instruction philosophy within a quasi-systematic model for mathematical concept building. The cognitive and affective development of high school students was studied within a collaborative, technology-based, socio-constructivist, self-guided environments, using a CAS calculator without traditional teaching or homework. Another research objective was to study the professional development of prospective teachers involved in the project. The studies utilized both quantitative and qualitative methods, including quasi-experimental pretest-posttest design and grounded theory.

The findings suggest that optimal 'student-centered learning' emphasizes students' freedom to choose learning objectives and working methods in problem-based socio-constructivist technology-based environments, in which open questions about both mathematics and technology are solved in collaboration between students or student teams. Even though students proceeded more or less chaotically, it was found that learning to link conceptual and procedural knowledge can be organized succesfully within a quasi-systematic framework. This paradigm allows integrate study modules of teacher education in new way. The term 'student-centered' should be redefined to minimize the role of the teacher as decision-maker with regard both to learning objectives and working methods. As this paradigm reflects how students work, communicate, and utilize technology outside the classroom, the focus should be shifted from well-tailored classroom lessons to students' free time activities. To use a motor racing analogy, the teacher becomes a team manager and the school serves as a pit stop. However, prospective teachers seem to be hesitant if this approach could replace the conventional teaching in school.

1. Introduction

This article basis of the thesis that presents a summary of studies triggered within the so-called ClassPad project among junior high school students, and later on being extended to studies among prospective teachers. The aims, methodology and findings, along with related articles, are summarized in Table 1. The columns constitute the main chapters of the article.

Adopting well-established criteria for the quality of research in mathematics education (e.g., [1],[2],[3]), "Topic realisation" indicates research significance, emphasizing the indicators gap bridging and research embedding (the extent to which the research is linked to previous studies). "Topic reconsideration" represents a new kind of research paradigm in terms of both aims and methods, stressing research novelty and originality, (the extent to which new knowledge is produced). In each case, these sub-chapters aim to ensure both the cognitive relevance and pragmatic relevance of the studies; research usefulness is addressed under the heading "Empirical evidence" and in the last main chapter, "Looking back and looking forward".

Instead of considering just one or two components of the process of teaching and learning at a time, the article seeks to address the working reality of that process in all its complexity. It might be argued that a summary article cannot fulfil the demands of rigor and precision in respect of how the empirical, theoretical and analytical aspects of the study are designed, implemented and reported.

Table 1. *Summary of the research*.

Chapter	From learning of mathematics to making of mathematics	Combining systematic planning and minimalism	Reappraising the term ‘student-centered’	Challenges for teacher education
Publication	A [5]	A [5]	E [9], D[8] and A[5]	B [6] and C[7]
ClassPad project phase	ClassPad 1 May–August 2005	ClassPad 2 October 2005–January 2006	ClassPad 2 October 2005–January 2006 and post- project	ClassPad 2 October 2005–January 2006, and post-project
Investigation aims	The influences of informal mathematics making on student mathematical identity	Quasi-systematic framework with minimalist instruction (MI) to monitor cognitive development	Student affective variables in technology-based learning environments with minimalist instruction Student experiences and expectations of mathematics lessons Shifts in student mathematical identity	Emergent paradigm of teaching and learning when prospective teachers plan their own mathematics lessons Development of student mathematical identity
Participants	Students at 8th grade (n = 15)	Students at 9th grade (n = 23)	Students at 9th grade (n=23) Students at 7th grade (n=17)	Prospective teachers (n=188), (n=116), and (n=66)
Data	Portfolios, questionnaires and interviews	Cognitive tests results, and student lesson diaries	Portfolios, written essays, questionnaires, and lesson observations	Written lesson plans, questionnaires
Methodology	Content analysis	Quasi-experimental and content analysis	Grounded Theory and content analysis	Theory driven content analysis Statistical analysis
Findings	Spontaneous mathematical activities with progressive technology can cause remarkable shifts in student mathematical profiles, opening new progressive ways to arrange the teaching and learning of mathematics.	Sophisticated interplay between systematic and minimalist approach can be achieved. Cognitive results in a paper and pencil test were significantly higher than those of students after traditional teaching.	A model to describe the process of mastering doing and learning mathematics through acquiring expertise. Two different learning profiles. The lack of autonomy could make weakness of students’ mathematical identity.	Prospective teachers have difficulties even identifying the pedagogical function of each task type in the ready-made learning material they tend to use for teacher-centered solo-learning. The integration of progressive technology as a part of students’ pedagogical thinking can shift student profiles
Outcome	New way of learning mathematics on free time.	New way of teaching mathematics in classroom.	New way of organizing student-centered learning environments.	New ideas for teacher education Need for shifts in school culture.

The validity, precision of meaning, and conclusions drawn from these studies are linked to predictability and reproducibility. By reading the related articles, it should, in principle, be possible to follow the research procedures and to replicate these studies.

As indicated in the Methodology row in Table 1, this article can be described as multi-method research. Beginning from qualitative and quantitative methods, the latter takes data from the quasi-experimental pretest-posttest design. More sophisticated qualitative methods were needed as the different phases of the ClassPad project generated more and more interesting questions. This variety of methods allowed simultaneous consideration of several factors affecting the process of teaching and learning.

It is appropriate to draw attention as well to certain limitations of this research. In particular, the number of subjects participating in the qualitative studies was small. However, it is typical of a grounded theory approach that even a small amount of data can be turned into a validated model if the analysis is done carefully. However, according to Thomas and James [4], for example, the procedural rules can make the grounded theory method an attractive option, in that it offers a map for conducting a qualitative inquiry. In these studies, the method was considered a suitable starting point for analyzing the students' experiences, enabling a model to be built in the ClassPad project of mastery in learning and doing mathematics.

2. From learning of mathematics to making of mathematics

Main outcome: Spontaneous mathematical activities with progressive technology, even of short duration, can cause remarkable shifts in student's mathematical profiles, opening new and progressive ways to arrange the teaching and learning of mathematics.

2.1 Topic realization

The international scientific community has acknowledged the global reach of problems in mathematics education (cf. e.g. [10]; [11], [12]). Possible reasons for the extent of the difficulty are summarized as follows by the Joint European Project *MODEM (*Model Construction for Didactic and Empirical Problems of Mathematics education)*:*

> *Mathematics tends to be explained as an organized body of knowledge, in which students are largely passive, practicing old, clearly formulated, and unambiguous questions for timed examinations. The large body of theory is found to be abstract and depends on an unfamiliar language. These features are of course essential for the purposes of a professional mathematician, but they leave many students dispirited and bored, and their performance in more advanced courses is poor because the foundations are weak: the examiners are reduced to setting only bookwork or stereotyped questions, which can be remembered without becoming a vital part of the student.*
>
> (http://wanda.uef. fi/lenni/modem.html)

When discussing the relation between mathematics and teaching of mathematics—and the poor reputation of both—Hvorecky and Haapasalo [13] adopt a business perspective, suggesting the following criteria for effective educational research:

- Each "market segment" has its own expectations; relevant priorities should therefore be established for different groups of pupils/students.
- As "the customer is always right", mathematics should be made more "edible and digestible" for each segment by bringing it closer to their environment and cultural values.
- Research in mathematics education should concentrate on specifying these particular needs and expectations by combining systematic planning with "minimalist" instruction (see Chapter 3).

In emphasizing the student's curiosity as a crucial motivational factor, Hvorecky and Haapasalo offer the following suggestions:

- Show the human character of mathematicians, who also make mistakes sometimes.
- Utilize modern progressive technology to visualize abstract objects and assign meaning to them.

- Formulate problems so as to stimulate students' interest in solving them.
- Make imperfect calculations for our imperfect world.
- Underline contradictions between naïve and formal solutions.
- Provide realistic models, including non-linear ones.
- Demonstrate the limited application of some models. [13]

The modern "science of knowing and learning" implies that we should teach less formal mathematics. Like Freudenthal ([14], [15]), we should see mathematics as a mental art, emphasizing active formulation and solving of problems. Beyond presenting the logical organization of mathematical knowledge, attention should be given to developing the student's ability to construct and understand knowledge and not merely to collect data. Zimmermann, Fritzlar, Haapasalo, and Rehlich [16] stress that

> *both technological applications and the history of mathematics offer excellent problem fields—already in elementary mathematics—for developing students' mental activities from the very beginning. Understanding the technological and cultural perspectives of mathematics is to our mind an effective preventative measure against students' negative beliefs about mathematics, poor self-confidence and inter-cultural contradictions (*p.214).

Kadijevich [17] points out four areas that have been neglected in mathematics education research: (1) promoting the human face of mathematics; (2) relating procedural and conceptual mathematical knowledge; (3) utilizing mathematical modelling in a humanistic, technologically-supported way; and (4) promoting technology-based learning through applications and modelling, multimedia design, and online collaboration. These findings set out the challenges in utilizing modern technology in all its forms.

During the last two decades, as in many other fields of education, the general paradigm of teaching and learning in mathematics has changed. First, there has been a change from the behaviorist knowledge transmission model to the constructivist paradigm, emphasizing cognitive processes. Second, there has been a shift to the socio-constructivist (socio-cultural) paradigm, emphasizing social factors in constructing shared knowledge ([18], [19], [20], [21], [22]). In research on mathematical instruction, the focus has been on collaborative problem-based methods for improving learning processes and outcomes ([23], [24], [25], [26]). However, the traditional direct teaching approach (cf. [27], [28], [29]) is still strong among teachers of mathematics, and the need to shift this culture towards student-centered approaches remains a topical issue ([18], [19]).

2.2 Topic reconsideration

What does it mean to "make mathematics"?

Regarding the first of the neglected areas mentioned above, Zimmermann ([30], [31) articulates eight sustainable motivations and activities (to be called *Z-activities*) that have regularly led to mathematical innovations across different times and cultures for more than 5000 years:

- *Apply*: application; applying mathematics; modelling
- *Construct*: history of practical geometry and architecture
- *Calculate*: repeatability and security of calculation methods (algorithms);
- *Play*: game simulations; gambling; playing
- *Evaluate*: aesthetics and aesthetical systems of values; evaluating; rites, religions and other systems of values; believing
- *Find*: interest in heuristics (methods of inventions); finding and inventing
- *Argue*: interest in methods of verification; proving and arguing
- *Order*: interest in systems and theories; axioms and ordering

We found it relevant to take these Z-activities (illustrated in Figure 1) as one element of a theoretical framework for structuring of learning environments, for analyzing students' cognitive and affective variables, and as a means of assessing the quality of teacher education.

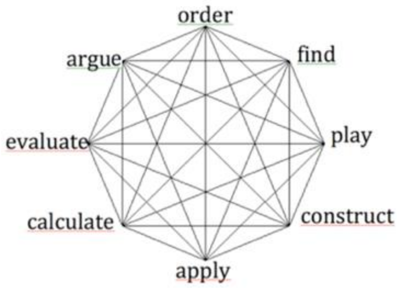


Figure 1. Activities and thinking tools which proved to be especially successful in producing new mathematics [31].

To examine how these eight activities emerge in students' working processes, we quantified each activity in terms of its distance from the centre of the octagon, indicating how well a student thinks he or she is doing in each activity, how strongly each activity is represented when using the term 'mathematics', and how well a computer can help to perform each activity. We found (see [32]) that school mathematics seems to provide little support for these activities in either Finland or Germany, and that university mathematics offers even less. The only exception is in respect of calculating, which seems overdone in every case. Using the same instrument, Haapasalo and Eskelinen [33] confirmed this outcome, suggesting that students seem to gain modest support for Z-activities from mathematics teaching in school, and from their overall usage of information and communication technologies.

Instrumental genesis

Zimmermann's analysis reveals that priority has often been given to the pragmatic aspects of mathematics. This was particularly true in ancient China, where there was much less interest in proving things than in ancient Greece, where the Euclidean and Archimedean traditions emphasized arguing, rules and using a conceptual structure [34]. In prescribing that mathematics should be taught both theoretically ('on paper') and in practice (such as 'actual surveying in the field'), the Dutch mathematician Stevin (1585) [35] suggested that tools such as rulers, compasses and right angles should be replaced by fieldwork tools. Nowadays, such tools often take the form of hands-on technology.

The development and use of information and communication technologies (ICT) can be understood as *instrumental genesis*, triggered by instrumentation and instrumentalization ([36], [37]). To describe the use of technology in education, Samuels and Haapasalo [38] suggest this term can, in a broad sense, be interpreted to mean the simultaneous acquisition of conceptual and procedural knowledge through the use of ICT. This is based on a view of technology that distinguishes between *artefacts* and *instruments*, referring to Vérillon and Rabardel's [39] distinction between an artefact as a physical object and an instrument as partly a physical object and partly a cognitive scheme enabling it to be used to perform specific kinds of task. Instrumentalization is directed towards the artefact and describes the process by which it becomes useful to the learner for specific purposes (that is, an instrument). Instrumentation is directed towards the learner and describes the process by which the possibilities and constraints of the artefact shape conceptual understanding and procedural ability ([36], [40]).

As derived through a form of sociocultural studies, instrumental genesis is deployed predominantly in the field of mathematics education in the use of technologies combining a *computer algebra system* (CAS) and *dynamic geometry software (*DGS). CAS can carry out algebraic operations and plot algebraic functions, and DGS allows geometrical objects to be drawn and constructed on the basis of geometric and algebraic definitions. Nowadays, most of those combined features are implemented in hands-on technology[1]. This may be one reason for Haapasalo's [41] view that

> *instead of speaking about 'implementing modern technology into the classroom' it might be more appropriate to speak about 'adapting mathematics teaching to the needs of information technology in modern society', and to shift from providing systematic instruction to a minimalist approach (see Chapter 3.2).*

This implies following Zimmermann's approach by giving greater emphasis to the making of informal and pragmatic mathematics rather than to formal mathematics.

Reappraising learning technology within radical constructivism

In their comprehensive analysis of learning technologies, Samuels and Haapasalo [38] conclude that

> *most so-called learning technologies bear little relation to the kinds of technologies contemporary learners use in their free time and thus appear alien to them and unlikely to stimulate them to informal learning. The theory debate behind learning technology has been characterised as a "see-saw" between instructivism and constructivism with additional emphasis being placed recently on the social context of learning. A contemporary approach to the use of some learning technologies in mathematics education potentially overcomes this see-saw by enabling the simultaneous acquisition of conceptual and procedural knowledge.*

The same authors suggest that learning technologies, at least for scientific subjects, should have the following attributes:

- *They should be innately stimulating while encouraging learners to increase their knowledge. In other words, they should put the curriculum into their fun (the kinds of experience learners find naturally enjoyable) rather than vice versa.*
- *They should allow learners to move from instrumentation to instrumentalization —that is, allow them to modify built-in knowledge to suit their particular needs.*
- *They should allow learners to construct conceptual knowledge that promotes applicable and viable procedural knowledge.*
- *They should either build in or facilitate appropriate scaffolding and assessment through external tutoring to avoid an entertainment or socialising orientation.*

In this article, the term *progressive technology* will be used to denote the combination of CAS and DGS. These tools are so powerful that even their developers seem less than fully aware of their huge potential for the learning of new concepts [42]. Across millions of pages on the Internet, examples are limited to how a given tool or application can be used for a specific operation. Even then, the user has to understand concepts beyond those operations as well as about the commands for use of that particular tool or application. This thesis proposes an opposite starting point: the user needs to know almost nothing about either the conceptual domain or the user commands. Adopting the view of Samuels and Haapasalo [38], the focus is on the immersive and interactive environments that the utilization of technology can create. The term *investigation space* is used in the constructivist sense to emphasize students' own freedom to find, identify, manipulate, and evaluate knowledge, and to solve problems collaboratively within one's own personal and social control. These characteristics of *navigationism* [43] include cognitive and psychological aspects [44], but also give due emphasis to the Z-activities.

[1] see e.g. http://edu.casio.com/products/classpad/cp2/ and http://education.ti.com/en/us/nspire-family/cx-handhelds.

As regards an appropriate socio-constructivist framework, Peirce's well-known *pragmatic theory of truth* offers a solid grounding, making the debate between radical and weak constructivism sound unnecessary and even naïve. Within this framework, once an investigation space has been designed (an open problem is given), the teams work in collaborative causal interaction with the problem. After testing the viability of radical ideas among and between the teams, only those ideas finally remain that are viable for the whole social group constituted by those teams (see Figure 2). The objectivity of knowledge relates to what the teacher and students consider necessary "to be able to cope", in the sense of von Glasersfeld [45].

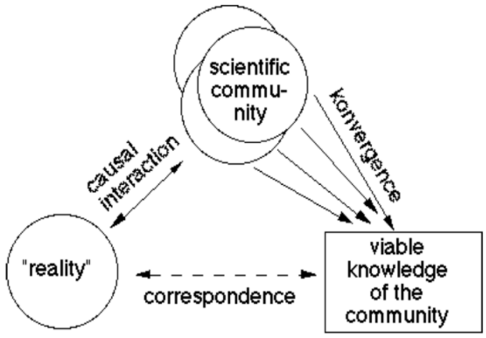


Figure 2. Viable knowledge as a result of radical social constructions [46].

Aims and methods of ClassPad 1

To establish whether students would choose to make spontaneous use of a CAS calculator, an unfamiliar tool (ClassPad 300) (see http://classpad.net/) was demonstrated briefly, with minimalist instructions (see chapter 3.2), to 8th grade students. They were offered the opportunity to play with it voluntarily during their summer holiday to learn concepts of 9th grade mathematics (such as a linear function). Their only required duty was to maintain a portfolio if they worked with the tool. Four students immediately volunteered for this study, and within a few days the number of volunteers had increased to fifteen. These students were given a leaflet that included information about some problems connected to the basic features of linear functions, which is one of the most important concepts in the 9th grade curriculum. This was the starting point for our ClassPad-project, which turned out to extend to three phases.

Given this reconsideration of the topic, it was important to design a new kind of instrument for studying students´ views and self-confidence with respect to mathematics and technology. For each of the activities in Figure 1, we quantified the distance from the centre on a 5-point Likert scale, developed to measure three profiles based on the Z–activities. This kind of instrument was subsequently used in all quantitative studies, using data from web-based questionnaires.

The research questions were designed to assess students on the following profiles:

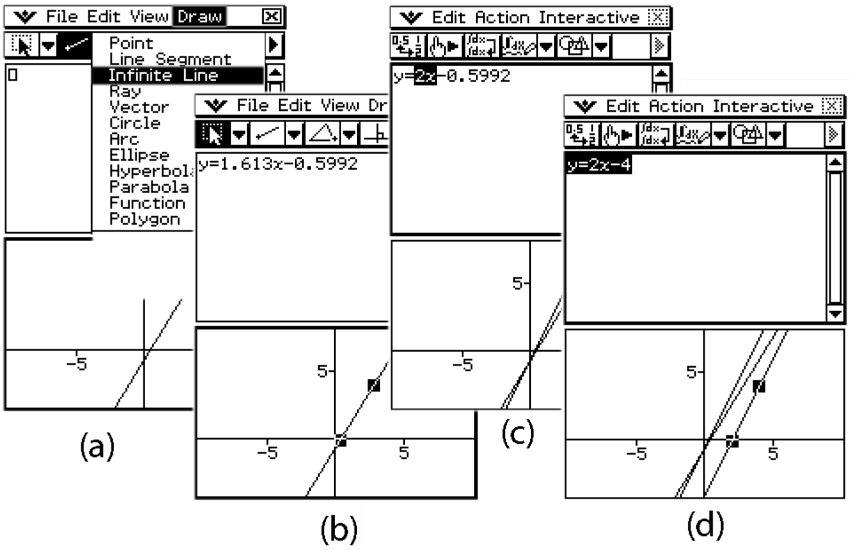
- *Math-profile*: how strong each activity appears when using the term 'mathematics'
- *Identity-profile*: how good the student thinks he or she is at each activity
- *Techno-profile*: How well a computer can help to perform each activity.

The quantitative data was consolidated by qualitative methods as individual interviews basing on the objects' working experiences (portfolios) and their responses to web-based questionnaire. After the summer holiday, the researcher conducted in-depth-interviews with each student (N = 15). The interview themes focused in particular on students' experiences of success in the project, their learning experiences, their own descriptions of their working process, and their self-evaluation in respect of the three profiles listed above. All the interviews were audio-recorded and then transcribed for analysis, varying in length between 20 and 30 minutes.

The analysis was based on content analysis, and the interpretative approach was phenomenological (cf. [47], [48], [49]). This method considers the characteristics of language as communication, focusing on the content or contextual meanings of text data ([50]). To begin, researchers worked through all the data. Then, the rough data was compiled according to the main themes above. Next, the thematic texts were categorized according to the emergent categories. Finally, the categories were generalized in search of answers to the research questions. In the following chapter some interesting samples from these interviews will be represented.

2.3 Empirical evidence

The sample below is taken from the portfolio of a relatively low-performing student (75 minutes work, beginning at 00:27).



6th session 15.07.2005. Time 00.27. (duration 1h 15 min)

- The equation of a line is y = 1.613x - 0.5992
- When changing to y = 2x - 0.5992 the angle between the line and y-axis is smaller.
- When changing to y = 1x - 0.5992, the angle between the line and y-axis becomes bigger.
- The equation is y = 1.613x - 0.5992.
- I change to y = 1.613x - 4, the line moves to the same direction away from origin.
- When changing to y = 1.613x + 4, the line moves to another direction on x-axis.

7th session 15.07.2005. Time 11.46. (duration 2 h 37 min)

- I try again, and now I do it with a little bit easier equation than yesterday.
- The equation is y = 2x - 3. I move the line to the origin.
- By changing the equation to y = 2x - 5 the line moves "forward to the positive x-axis".
- When changing to y = 2x - 1, the line goes to the same direction with amsller the distance.
- It was something like this -> Amazing…

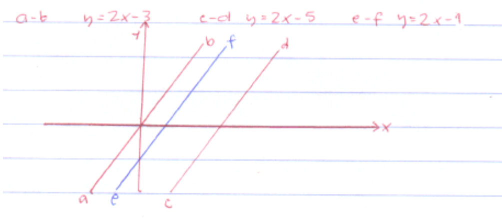


Figure 3. Portfolio example of utilizing simultaneous activation when playing with ClassPad [5].

This portfolio example contrasts with the common purely metacognitive abilities of students and teachers [41]. By manipulating the equation (conceptual interpretation) spontaneously, the student explained how the parameters affect the position and location of the line (procedural interpretation). She made through instrumentalization her own interpretation against the standard view: the line moves along the horizontal axis. Figure 2 represents the profile shifts of this same student. The profiles shifted towards the creative direction, the biggest change being in *playing*. In the interview, the student said, “In May I could not even think to play with the ClassPad in the summer holiday. However, I noticed, that it was very capable for playing with mathematics.” Maybe the most surprising shift occurred in the *calculate* dimension. Here, the student became aware of the versatility of the technical tool, which was connected with a decrease of the relative intensity of the belief on the importance of calculations.

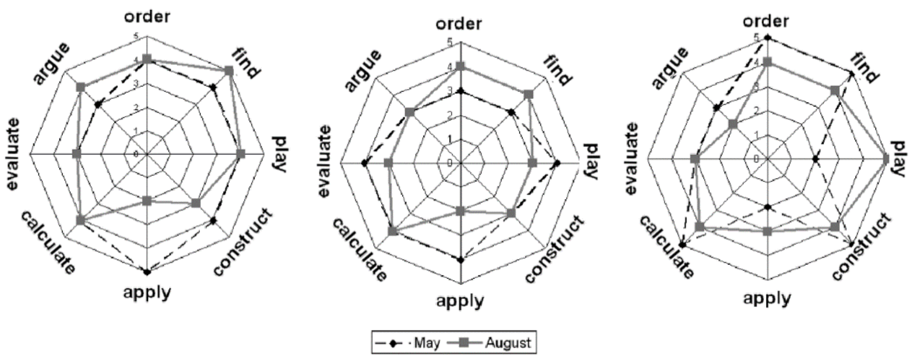


Figure 4. Shifts in the student's Math-profile (on the left), Identity-profile (in the middle), and Techno-profile (on the right). [5].

3. Combining systematic planning and minimalism

Main outcome: Students’ cognitive development can be supported by socio-constructivist minimalist instruction provided that learning environments are planned in a research-based, quasi-systematic way to link conceptual and procedural knowledge.

3.1 Topic realization

Characterization of two knowledge types

The characterization of two knowledge types has been identified as a neglected area in mathematics education research— even though it is a key question for any pedagogy to ask whether the learner must understand before being able to do, or vice versa. This means solving the conflict between conceptual and procedural knowledge, characterized according to Haapasalo and Kadijevich [53] as follows:

- *Procedural knowledge denotes dynamic and successful use of specific rules, algorithms or procedures within relevant representational forms. This usually requires not only knowledge of the objects being used, but also knowledge of the format and syntax required for the representational system(s) expressing them.*
- *Conceptual knowledge denotes knowledge of particular networks and a skilful “drive” along them. The elements of these networks can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may introduce a new concept or rule) given in various representational forms.* (p.148)

When designing any learning environment, we encounter a conflict between these two knowledge types. Samuels and Haapasalo [38] emphasize that when scaffolding operational learning within scientific subjects, one of the key questions is how to promote acquisition of both conceptual and procedural knowledge while avoiding any polarization between recipe-orientated, methods-based teaching and an abstract conceptual approach. Implementing technology often makes this still more complicated by requiring a dual-focus approach: a well-balanced convergence between instructional orientation and the demands of the technology. Procedural skills alone are insufficient for a transfer effect, if logic and significance beyond those skills is unknown. On the other hand, a solely conceptual understanding of an application (e.g technical tool or software)—retrieving required information from memory and interpretation to implement concrete procedures—can be difficult. If the application to be learned is simple in nature, conceptual training can lead to awkwardness in use of the application. Resolving these issues calls for a sophisticated interplay between conceptual and procedural knowledge of both the topic to be learned and how the technology is to be used. If it is agreed that the main goal of education is to develop both procedural and conceptual knowledge and to make links between the two, the first crucial question is, what is the quality and the role of technological application? And secondly, how do different technologies and pedagogical solutions affect the relationship between the two knowledge types? According to Rittle-Johnson and Koedinger [51], conceptual and procedural knowledge seem to develop iteratively, in that a change of problem representation influences the relationship between them. This kind of development was verified through several empirical studies within the pedagogical model developed in the MODEM project (see [52], [46]). This model of the interplay between the two knowledge types makes use of spontaneous procedural knowledge as well as the simultaneous activation of conceptual and procedural knowledge.

Interplay of conceptual and procedural knowledge

Recalling the discussion of Haapasalo and Kadijevich [53] about the importance of procedural knowledge in human constructions of meaning, investigation spaces should allow learners to start from their spontaneous procedural knowledge. However, procedural knowledge alone cannot predominate if we consider that the main goals of education are to promote skilful navigation in knowledge networks, and the ability to apply knowledge in new situations, requiring linkage between the Z-activities. Recent research by Lauritzen [54] with economics undergraduate students (n = 476) reveals two crucial factors in acquiring and applying knowledge. First, procedural knowledge is necessary but not sufficient for conceptual knowledge; and second, to be able to apply what they know, students need conceptual knowledge. Combining these demands, we can conclude that the so-called *developmental approach*, based on a genetic view emphasizing procedural knowledge, needs to be combined with an *educational approach*, based on dynamic interaction and emphasizing conceptual knowledge (see [53]). Figure 5 shows a sophisticated interplay of the two approaches within the MODEM theory.

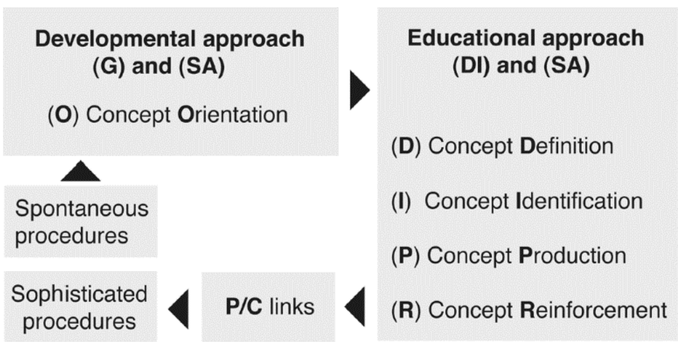


Figure 5. Interplay between developmental and educational approaches [46].

Orientation (O) forms the first phase of systematic concept building, utilizing what is basically a developmental approach, in which interpretations of the situation are based on the mental models of the pupils, from their more or less naïve procedural ideas. These act like a wake-up voltage in an electric circuit that triggers another much more powerful current, and is amplified again. The procedural and conceptual knowledge types start to support each other, affording an opportunity to use, for example, the principle of simultaneous activation (SA). The examples in Figures 6 to 8 are taken from the ClassPad 2 project.

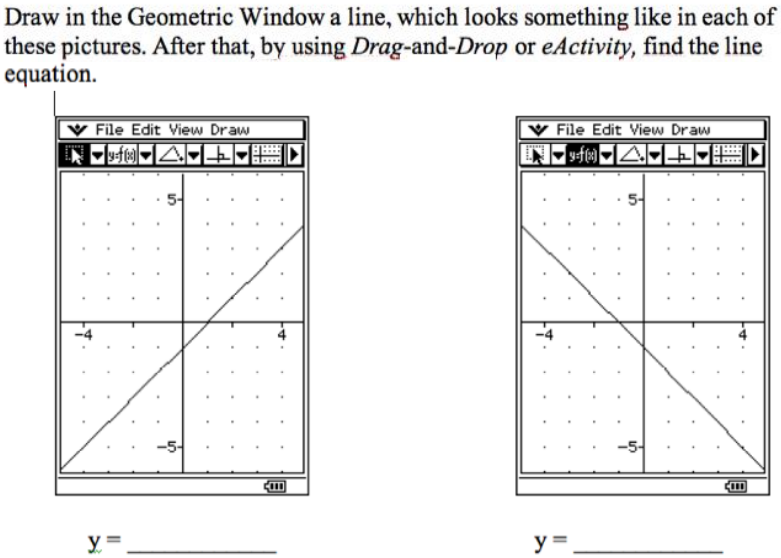


Figure 6. An investigation task with ClassPad at the phase of concept orientation.

The role of the concept *Definition (D)* is to give students opportunities to make their own investigations, to express the results of their investigation (especially in verbal form) in each case, and to argue about these results within and between the collaborative teams. By means of social construction, a definition for the concept is born, as students try to fix the relevant determiners of the concept in verbal, symbolic, and graphic forms. Creative thinking and productive work is needed, especially in the phases of orientation and definition. The next phases of concept building utilize the principle of dynamic interaction (DI); the idea is to give students a sufficient number of opportunities to construct concept attributes and procedural knowledge based on them.

In the phase of *Identification (**I**)*, students are encouraged to train themselves in recognizing concept attributes in verbal *(V)*, symbolic *(S)*, and graphic *(G)* forms. For this, we need six kinds of *(I)* tasks: ***I**VV, **I**VG, **I**VS, **I**GG, **I**SS,* and ***I**SG.* During the learning process, the teacher must be prepared, if necessary, to begin with tasks requiring only binary distinctions (between two elements), before advancing to the identification of several elements. Figure 7 illustrates task types ***I**GS* and ***I**GG.*

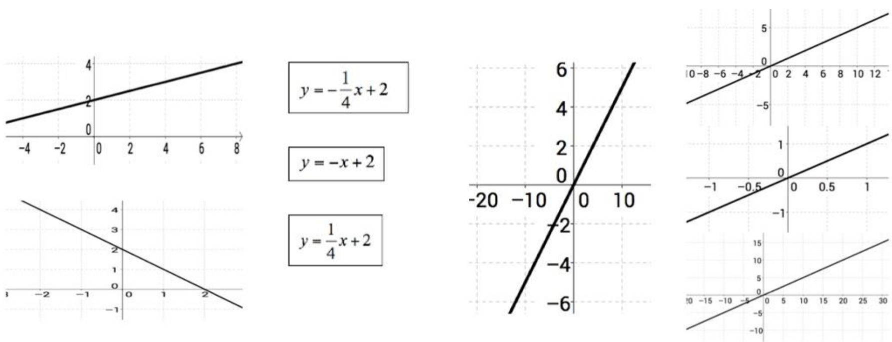


Figure 7. Identification task IGS (on the left) and IGG (on the right).

In the phase of *Production (**P**)*, pupils must be allowed to produce another representation in a different form from a given presentation of the concept. The development of production *(P)* requires nine combinations: ***P**GV*, ***P**GS*, ***P**GG*, ***P**SG*, ***P**SV*, ***P**SS*, ***P**VS*, ***P**VV*, and ***P**VG*. The tasks of identification and production must be capable of completion without any complicated processing of information on the student's part. Figure 8 represents task types ***P**GG* and ***P**GS* (even though the tasks can be solved, like other task types, by using *Drag-and-Drop* activity or *eActivity*).

Draw the line as represented in the above figure and find the line equation.

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Then, change the line equation so that the line goes like in the figure below.

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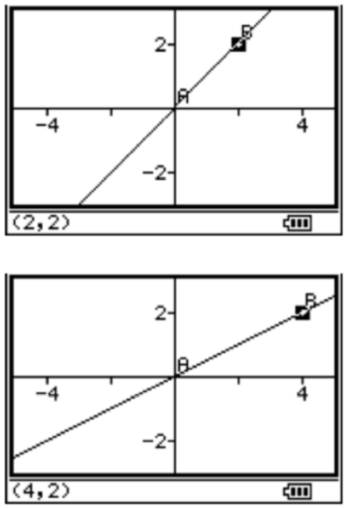

Figure 8. Production tasks PGG and PGS.

In the phase of *Reinforcement (**R**)*, the goal is to train and utilize concept attributes, and to develop procedural knowledge to be used in problem-solving and applications. The tasks covered in the next chapter can be classified as belonging to this category.

3.2 Topic reconsideration

Combining systematic planning and minimalism

The genesis of heuristic processes, and the ability of students to develop intuition and mathematical ideas within a constructivist or minimalist approach, is unlikely to be attainable without thorough planning of learning environments by the teacher. To this end, empirically tested and more or less systematic pedagogical models can be helpful. On the other hand, encouraged by the findings of Kadijevich and Haapasalo [55] that P-C links may be established by means of sophisticated conceptually-oriented technology-based environments, and our findings that students were able to use the tool on the level of instrumentalization, we orchestrated the learning of a whole 9th grade mathematics class using only the ClassPad calculator and without any textbooks or traditional homework (referred to later as instrumental orchestration). The learning tasks were designed within the quasi-systematic MODEM framework described above.

Recalling Figure 1, it is the right-hand half of the octagon that emphasizes creative human activities, which very often run optimally without any external instruction or requirement. Students frequently neglect teacher tutoring or they feel they do not have time to learn how to use technical tools; similarly, teachers feel they do not have time to teach how these tools should be used. The term *minimalist instruction* (MI), introduced by Carroll [56] is crucial, not only for teachers but also for those who write manuals and help menus for software. Carroll observed that learners often avoid careful planning, resist detailed systems of instructional steps, tend to be subject to learning interference from similar tasks, and have difficulty recognizing, diagnosing, and recovering from their errors. The assumptions, characteristics, and methods of minimalism implemented here are opposite to those of Gagné [57]. With a view to fostering problem-solving abilities, we pick up some alternate characteristics of MI (cf. [58]). These features of *minimalism* encompass several varieties of the constructivist view, and include certain assumptions about effective instruction (cf. [59]): 1) specific content and outcomes cannot be pre-specified, although a core knowledge domain may be specified; 2) learning is modelled and coached for students with unscripted teacher responses; (3) learning goals are determined from authentic tasks stressing doing and exploring; (4) errors are not avoided but used for instruction; (5) learners construct multiple perspectives or solutions through discussion and collaboration; (6) learning focuses on the process of knowledge construction and development of reflexive awareness of that process; (7) criteria for success are the transfer of learning and a change in students' action potential and (8) the assessment is ongoing and based on learner needs [41].

Aims and methods of ClassPad 2

The findings from ClassPad 1 encouraged us to investigate whether students returning from their summer holiday were able to use the tool on the level of instrumentalization in the classroom. We therefore planned and implemented learning of the core areas of 9th grade mathematics—linear functions and basic concepts of statistics—using only the ClassPad calculator, without any textbooks or traditional homework. The learning of linear functions consisted of nine 45-minute lessons. During the first lesson, the students formed teams and learned to use ClassPad. The focus was on changing representations between algebraic and geometric calculator windows by utilizing the principle of simultaneous activation of conceptual and procedural knowledge (cf. portfolio sample in Figure 3).

The research questions were:

> Q1. Can a quasi-systematic framework to link conceptual and procedural knowledge be used within a minimalist approach to instruction?
>
> Q2. Which kinds of cognitive development can be found among students?

Regarding Q1, the learning tasks were planned according to the systematic MODEM framework, comprising 8 tasks for concept Orientation and Definition, 13 tasks for Identification, 20 tasks for Production, and 13 tasks for Reinforcement (see Figures 6 to 8). The students had complete freedom to choose a problem set from the menu, and to decide how they would work during the lessons. Students' choices were recorded by prospective mathematics teachers (PMT) who participated in the project within their pedagogical studies.

Regarding Q2, the students' cognitive skills were tested three times: a pre-test before, a test soon after, and a delayed post-test five months after the learning period. The tests were designed according to the MODEM framework to fit the test used in the MODEM 1 study [60]. This allowed comparison of the cognitive results with those gained from cognitive mathematics teaching. This kind of test design, using a single group of participants without a control group, was considered useful for describing group development during the process ([61], [62], [63]).

3.3 Empirical evidence

Interplay of systematism and minimalism works

Our results suggest that the answer to research question Q1 is affirmative. Figure 9 presents an example of how one of the student teams chose tasks from the menu, in a far from optimal way. The team went directly to ***PSG*** tasks and also selected from that list those eleven tasks (#1–#11) more or less randomly (as was the case for the Orientation Tasks). Students evidently liked the amazing drag-and-drop function, which automatically performed the ***PSG*** action.

→	***P***SG (#1)	→	***P***SG(#5)	→	***P***SG(#3)	→	***P***SG(#7)	→	***O*** (#3)
→	***O*** (#1)	→	***O*** (#2)	→	***P***SG(#2)	→	***I*** VG(all)	→	***P***SG(#6)
→	***I*** VS(all)	→	***I*** SS(all)	→	***R*** (#1)	→	***R*** (#1)	→	***R***(#1)
→	***P***SG(#4)	→	***P***SG(#9)	→	***P***SG(#8)	→	***P***SG(#11)	→	***P***SG(#10)
→	***P*** VV(all)	→	***P*** GV(all)	→	***P***SG(#12)	→	***P*** GG(all)	→	***P*** VV(all)

Figure 9. An example of a "classpath" when selecting tasks.[5]

Another team started by picking up a quite complicated problem on optimizing mobile phone costs, which was planned as a Reinforcement task. After realizing that the (partly linear) cost models was too difficult for them, the team then chose a different, and much easier, problem set of Identification tasks—the lowest level of understanding of links between representations. This example shows that interplay between a systematic and minimalist approach can be achieved even by simple pedagogical solutions.

Positive surprises in students' cognitive development

Students' scores in all test items except application task #5 were significantly higher ($p < 0.05$, sign test) after the working period than in the pre-test (see Figure 10). The most remarkable changes ($p < 0.001$) were in task types (PVV), (PSV), (PVS), (PSS), and (IVV). For most of the test items, these students' scores were higher than those of students even at the end of junior high school, after gaining conventional mathematics teaching, revealed by the MODEM 1 study (see [52], [60]). Furthermore, these students' scores (especially in the production tasks) were even better in the post-test [64]. The results suggest that students learned the concept of a linear function, if "learning" is defined as in previous studies ([60], [65], [66], [67], [68], [69]) The positive results somewhat contradict the results of Kirschner et al. [70] that minimally guided instruction is less effective and less efficient than other instructional approaches (cf. [71]). In this light, we wanted to extend the methodology to find out what had actually happened during the learning processes, and what might have contributed to the positive results. These issues will be discussed in chapters to come.

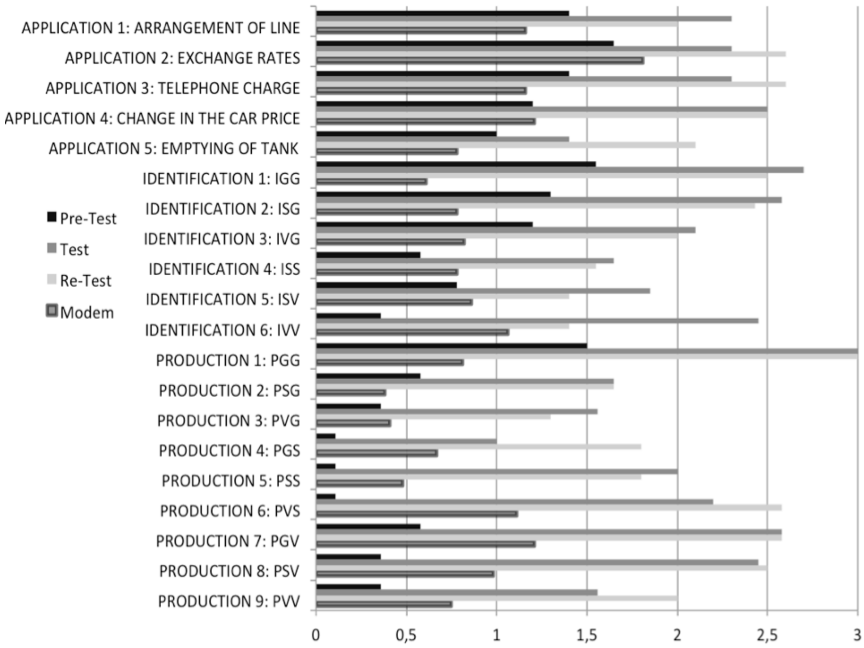

Figure 10. Students' scores in the tests, compared with test scores in MODEM 1 study.

4. Reappraising the term ’student-centered’

Main outcome: Instead of concentrating ON each student at a time, the focus should be shifted to doing and sharing WITH students and student teams.

4.1 Topic realization

Student-centered learning (SCL) is a complicated, nuanced, and messy concept. O’Neill and McMahon [72] describe the interpretation of the term “student-centered learning” as appearing to vary between authors. Some equate it with active learning, while others subscribe to more comprehensive definitions such as choice in learning or exchange of power in the teacher-student relationship. SCL has also been defined ([73], [74]) as the conceptual opposite of teacher-centred learning ([75], [76]). According to Zimmerman [77], that means the teacher presents information in the hope that students will take it in. By contrast, the SCL literature commonly emphasizes student autonomy and collaboration ([78], [77], [79], [80], [73]). According to Alliance of Excelent Education (AEE) [18], SCL has several characteristics that help to prepare students for college and a career, in that it is rigorous and based on real-world expectations; personalized; collaborative, relevant, and applied; and flexible, as learning can happen anytime, anywhere. To encompass these varying definitions of SCL, Neumann [81] presents a framework based on Bollnow’s [82] work, holding that student-centered learning contexts can be said to center *in* students, *on* students, or *with* students. Learning contexts that are centered *in* students imply that learning dwells within the student and is generated with little or no assistance from anyone else. On this view, educators simply need to not inhibit learning, which is seen as essentially self-generating and self-driven. Learning contexts that are centered *on* students, on the other hand, are “analogous to a kind of handicraft work.” [81]. In this context, teachers determine and provide for educational needs, and students react to those plans and prerogatives. Here, the focus is on a teacher’s ability to convey the necessary material, and on a student’s responsibility to learn it. Finally, learning contexts centred *with* students bring the teacher and the student into partnership as autonomous entities, implying that learning emerges through this collaboration.

SCL is also linked to classroom emotional climate (CEC). According to Hamre and Pianta [83], classrooms characterized as high in CEC have (a) teachers who are sensitive to students’ needs; (b) teacher–student relationships that are warm, caring, nurturing, and congenial**;** (c) teachers who take their students’ perspectives into account; and (d) teachers who refrain from using sarcasm or harsh disciplinary practices. In such classrooms, the teacher fosters student comfort and enjoyment by regularly expressing warmth toward, respect for, and interest in students, and by encouraging their cooperation with one another. In classrooms high in CEC, teachers are also aware of their students’ emotional and academic needs, and respond by choosing age-appropriate activities that encourage self-expression and cater to their students’ interests and points of view. Using multilevel mediation analyses and controlling for teacher characteristics and organizational and instructional climate, Reyes et al. [84] showed that the positive relationship between CEC and student grades was mediated by engagement. Effects were robust across grade level and student gender, confirming the role of classroom-based, emotion-related interactions in promoting academic achievement.

In exploring students' experiences of mathematics, there will always be background variables such as teaching arrangements, the relevance of teaching, or the influence of school culture. The first two of these variables are at issue in discussing the shift of paradigm towards socio-constructivism, where teachers’ actions are guided by SCL and achievement of a good emotional climate in the classroom. With regard to school culture, sociologists (e.g., [85], [86], [87]) have proposed three interlinked strata, dividing the school into physical, formal, and informal dimensions. Physical school includes space and corporeality, and the school environment effectively becomes the stage on which the formal and informal elements of school communicate. Formal school is determined by the administrative and educational purposes of the curricula, teaching materials, methods and classroom work, and learning activities and interactions, as well as by school rules and formal hierarchies.

Informal school includes education outside the school premises and interaction outside the classroom. The physical school is seen as an arena, in which students negotiate social positions through storytelling. Young people constantly monitor the finest details of the school environment, making judgments about how to be a good citizen, or a good student. While the formal school obliges teachers to supervise students and to realize learning objectives, the informal school and its alternative “rules” inform everyday school life, as discipline and student resistance co-exist in a repeating pattern [87]. Students' informal learning experiences are, however, constantly ignored by the learning processes of formal school, even where those experiences might offer a meaningful basis for learning [88].

In their comprehensive analysis of physical and virtual learning environments, Haapasalo and Samuels [44] emphasize that

> *learning is becoming more collaborative and mobile and today almost every student owns not only a mobile phone but also many otherpersonal devices. To get these devices integrated with mathematics is an interesting task, especially because recent studies show that student seem to learn mathematics in most effectiveway on their free time. Numerous studies show that enjoyment and creativity are seen as the key to affective motivation leading to an identity and attitude change towards mathematics.*

4.2 Topic reconsideration

Reappraising the terminology

To begin to reappraise the term ”student-centered learning”, the first requirement was to take a critical look at the use of the term ”learning”—which in itself is always essentially ”student-centered”. It seems strange to even suggest that someone is learning on behalf of another person, and it might be more appropriate to speak, for example, about ”student-centered teaching”, ”student-centered instruction”, or ”student-centered learning environments”. With this in view, the first step here was to identify the learning paradigm underlying classroom instruction. To this end, Silfverberg [89] (cf. [19]) posed two questions: (1) Who are the real actors in this process and what is the degree of collaboration? (2) Who are the primary decision-makers in this process? Based on these questions, we designed a matrix [6] for analyzing the lessons plans made by prospective elementary teachers (Table 2). The upper right-hand corner of the matrix can be seen to represent an ideal paradigm for orchestration of “student-centered learning environments” (SCLE).

Table 2. Matrix for student-centered learning environments (SCLE) categorized by two main characteristics: collaboration and decision-making (modified from [6]).

	NON-COOPERATIVE ON	COLLABORATION INSIDE A TEAM IN	COLLABORATION BETWEEN TEAMS WITH
MINIMALIST INSTRUCTION	Learning results and methods are determined by the student based on authentic problem-solving environment	Learning results and methods are determined by the student team within collaboration	Learning results and methods are determined through collaboration between the students teams
STUDENT-ORIENTED	Learning goals are mainly given by the teacher; the student decides how to interpret them and how use the material	The learning goals are mainly given by the teacher; decision to interpret them, and use of learning materials, is made collaboratively within the team	The learning goals are mainly given by the teacher; decision of interpreting them and using the learning material is made through collaboration between the teams
TEACHER-CENTERED	Teacher decides the learning material to be used within exact instructions	Teacher decides the learning results, gives the learning material to the team, and gives exact instructions for co-operation inside the team	Teacher decides the learning results, gives the learning material to the teams, and gives exact instructions for co-operation between the teams

Student centered learning enviroments (shaded area: minimalist instruction and student-oriented rows)

SCLE during ClassPad 2 characterized by student's experiences

Figure 11 illustrates how the matrix developed as the author who was also the teacher of that class was planning and realizing the work during ClassPad 2. The next chapter will consider how this view aligns with students' experiences.

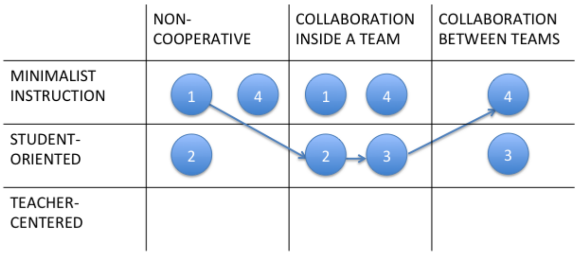

Figure 11. Stages of students' work during Classpad 2.

During the first introductory lesson (stage 1), the students formed teams and learned to use ClassPad without any written tutorials. They could ask help from the teacher or from their classmates, and the focus was on learning to utilize the drag-and-drop function to switch between the algebraic and geometric windows. Students could begin with any geometric figure they wanted, but the problems concerned only the straight line. Some of the students moved to stage 2 at the end of the first lesson; the others achieved this during the second lesson when they made decisions to use the problem menu to test drag-and-drop. At this point, most of the students were working in pairs. Stage 3 was reached when students tried to solve the problem in collaboration with the other teams. In this phase, team working varied between co-operation and collaboration. Stage 4 was reached when some students noticed that ClassPad is very useful for polynomials. At this stage, ways of working were still varied.

To find empirical support for the characterization of SCLE in Table 2, students' working processes during ClassPad 2 were looked from four different points of view: (1) How did students experience SCLE? (2) How did they communicate in SCLE? (3) What kind of influences did SCLE cause to students' mathematical identity, and (4) How does SCLE relate to students' expectation towards mathematics teaching?

Data for questions 2 and 3 was analysed using content analysis, and for questions 1 and 4 by using Grounded Theory method. Students wrote essays on their experiences, to be analysed using the "bottom-up" methods of grounded theory as described by Glaser and Strauss [90] and Glaser [91] (which introduced a more flexible approach). Quite rich data were obtained from the students' personal essays or reports, and three analytic steps were then followed. In the open coding phase, the data derived from the students' writings were examined in detail and coded for emerging concepts. During this open coding, concepts were identified. In the theoretical coding phase, the number of concepts was reduced and grouped into tentative categories. Patterns of students' processes were identified in the data, and the emerging categories and their relationships were carefully compared to ensure that these categories covered most of the variation found in the data.

4.3 Empirical evidence

SCLE during ClassPad 2 characterized by student's experiences

According to Eronen & Kärnä [9], students' experiences can be described by a model comprising four categories and contributing to a fifth category (Figure 12).

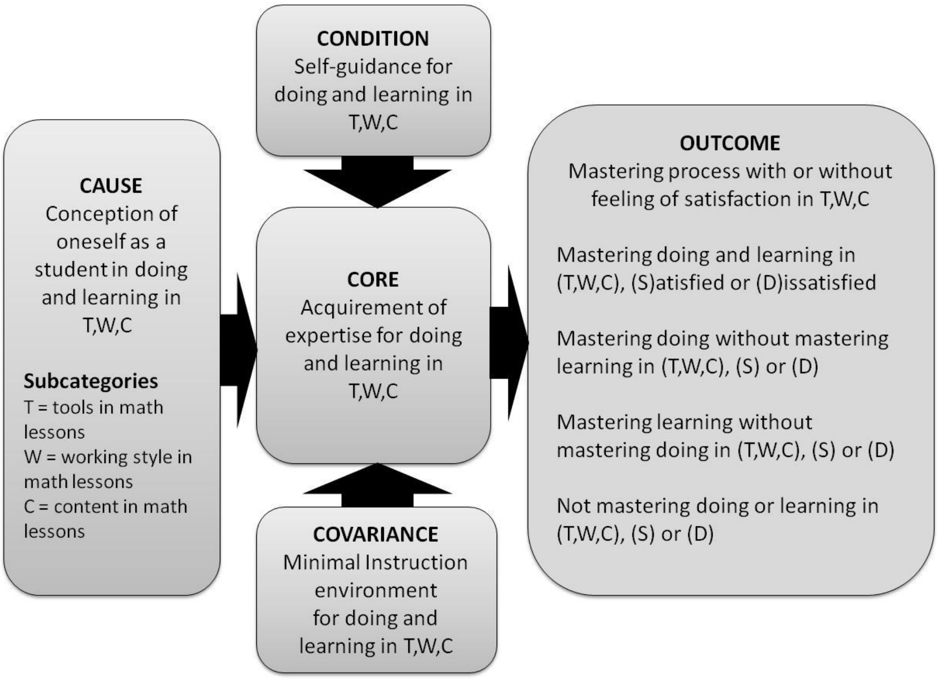

Figure 12. Model of mastery: doing and learning mathematics [9].

The model describes the processes through which students acquire the expertise needed for mastering doing and learning mathematics. Students' conceptions of themselves as math learners (Cause) determined the process needed for mastering doing and learning math. The process was shaped by the teacher's minimalist instruction (Covariance), and by the students' as self-guided and collaborative learners (Condition). Technology as a tool and problem-solving as content were additional important elements of the process. Thus, the learning processes during lessons were far from traditional and became student-centered. As the model shows, mastering doing and learning mathematics was the outcome of a complicated process shaped by a combination of factors: the students' conceptions of themselves as math learners; self-guided work style; minimalist instruction as a teaching strategy; and expertise in the project's new technical tool, work style, and math content. The process was iterative and included shifts between modes. The two typical student processes can be described respectively as the V and L profiles (Figure 13).

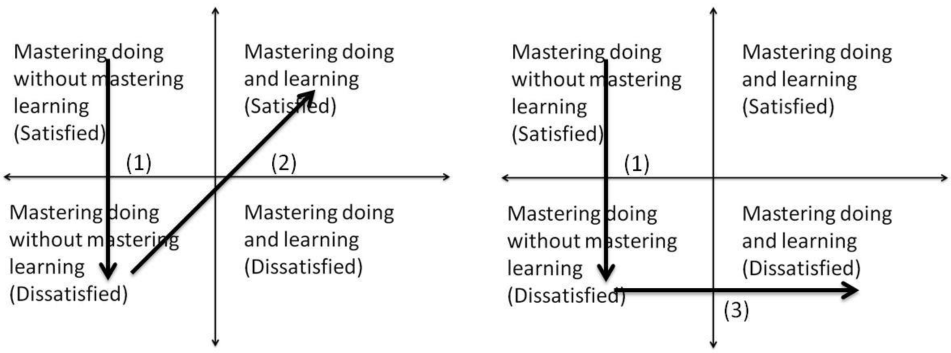

Figure 13. V-profile (left) and L-profile (right) for mastering doing and learning mathematics [9]

The more common process of mastering doing and learning math for most of the students (n = 21) is represented by the V profile. These students acquired expertise and were satisfied with their performance in all three subcategories (tool, work style, and content). Typically, the students' process for mastering these categories was as follows.

At the beginning (from stage 1 to stage 2, see Figure 11), the students were excited about this way of learning mathematics, which was very different from regular lessons, as students were asked to use a new tool, and to complete self-guided work. After the first few lessons, however, the satisfaction level of V-profile students collapsed (see (1), Figure 13), as they faced difficulties in mastering a tool and work style that slowed down the process of solving the assigned math problems. This satisfaction collapse is traced in the transition from stage 2 to stage 3 (Figure 11). As the project continued, these students acquired greater expertise in the tool and subsequently in the work style. Expertise in the content, that is, solving the assigned math problems, proved most difficult to acquire, usually taking several lessons. As the V-profile students came to understand for the first time how they could self-guide to solve problems by using the calculator, acquiring content expertise accelerated with increasing awareness of how to use the new tool and work style for learning math, and they moved from stage 3 to stage 4 (Figure 11). The final outcome of the process for the V-profile students was that they mastered doing and learning math and felt satisfied (see (2), Figure 13).

The learning process of two other students is described by the L-profile. These students mastered doing and learning math but were dissatisfied at the end of the project. At the outset, both were as excited as the V-profile students. However, as the project progressed, the L-profile students had difficulty acquiring at least one of the three fields of expertise. One of the students acquired expertise in the new tool and work style, but did not acquire content expertise—that is, completing the problems assigned for the project. This can be explained by his decision to work alone. He reported that some of the tasks were easy to solve using ClassPad, but that some tasks were too difficult to solve alone. The other student also decided to work alone. In like manner, she did not acquire content expertise. She reported that, although she completed all the math tasks, she learned almost nothing during the project because of the new work style. However, she also mentioned that she had learned the concept of slope, which was crucial in understanding the equation of a straight line. To this extent, she had acquired expertise in the tool and content, but did not feel she had mastered the work style. The outcome of the project for these L-profile students was that they mastered doing and learning math while being dissatisfied. Even if these students had mastered doing and learning math, they were dissatisfied if their learning process was guided by unilateral cooperation with peers or the teacher (see (3), Figure 13). It became evident that, for these two students, completion of stage 4 processes would require effective collaborative problem-solving.

Student communication in SCLE

Figure 14 represents an instance of student communication during one lesson, and is quite typical not only of these students but of the majority of students. The analysis revealed that the dominant category of communication was *entertainment*. This lesson was very typical for students throughout the project. Both students went for Production tasks 1 to 4 during the first 19 minutes (cf. Figure 14). Their communication during the first six minutes concerned just entertainment or their daily school activities. After that, both were silent for about one minute until Sarah started to communicate for three minutes to have entertainment. Figure 11 shows that the amount of discussion then decreased quite radically. We formed the view that this was caused by the difficulty of Production tasks that were not appropriate for this low-performing pair. The fact that after 23 minutes John stayed with Production Task #7 whereas Sarah attempted tasks #5 and #7 also indicates poor collaborativity. After that, the teacher took more than 10 minutes to explain how to utilize the eActivity, and the Drag-and-Drop properties of ClassPad. Students then selected Orientation Task #1 from the menu, and after that Identification Task ***IVS***. However, the discussion kept returning to technological aspects or formal entertainment rather than mathematical content. The recording of this lesson (41 minutes) includes discussion of content (6 minutes), ClassPad (6 minutes), and studies in general (6 minutes), with 3 minutes of silence. The dominant category of speech (21 minutes, more than 50% of the total) was entertainment. This example demonstrates the significance of informal school when students work within SCLE, as well as the importance of storytelling and the potential of informal school culture activities as part of the learning process (cf. [87]).

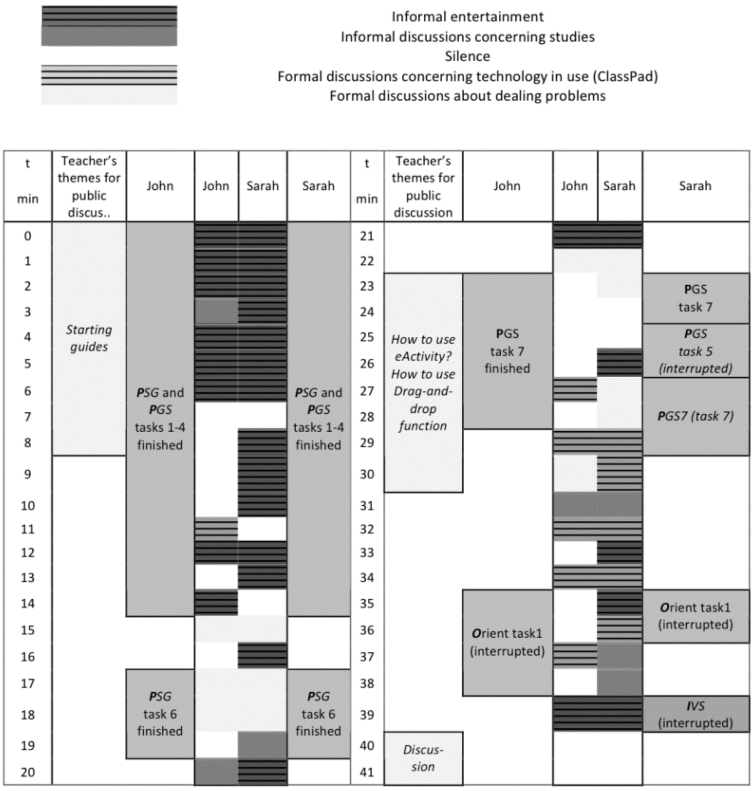

Figure 14. A sample of communication flow within a student pair [92].

Shifts in students' identity profiles and math profiles

Turning now to another interesting example from the ClassPad 2 project, it becomes clear how Identity-profiles and Math-profiles (see Chapter 1.2) change when students work in pairs. In this instance, a conceptually-oriented peer-teacher (i.e., wanted to know what steps she has to do) was teaching her procedurally-oriented classmate (i.e. uninterested in understanding what he was doing; see [93]). Figure 15 shows that this kind of peer-teaching period quasi-enriched the Math-profile of the peer-teacher, but undermined her mathematical self-confidence. Interestingly, the Identity-profiles of these two students seem to run in opposite directions. As the classmate begins to think (perhaps wrongly) that he can find, apply, and argue better than at the outset, his peer-teacher's own self-confidence in making mathematics seems to deteriorate (see Figure 16). As the peer-teacher spent all her time in explanatory mode, this finding may indicate that a behaviourist approach to teaching can damage both student and teacher.

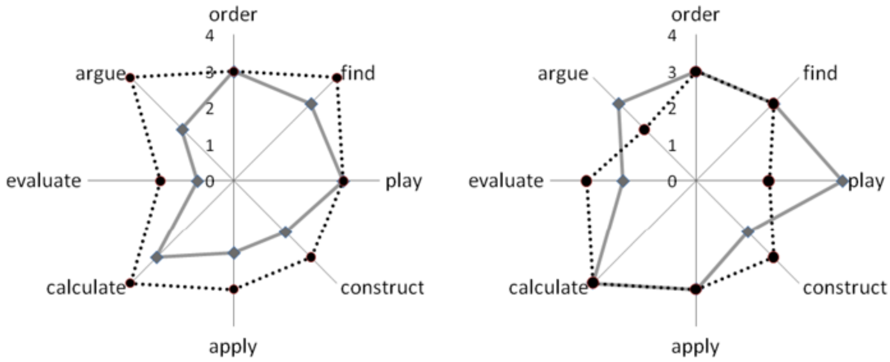

Figure 15. Math-profiles of the peer-teacher (on the right) and her classmate (on the left) at the beginning (dashed) and at the end of the working. [5]

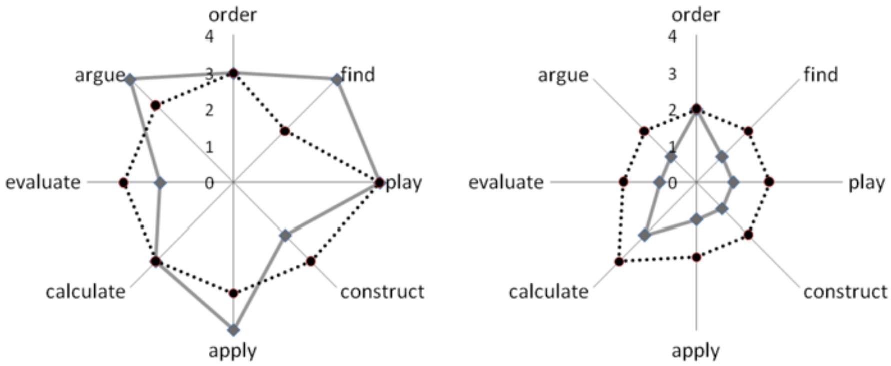

Figure 16. Identity-profiles of the peer-teacher (on the right) and her classmate (on the left) at the beginning (dashed) and at the end of the working. [5]

Students' expectations about mathematics lessons

It was also important to ascertain what kinds of expectations students have about mathematics lessons, and how they are linked to SCLE. To this end, we asked 7th grade students to describe the elements that make mathematics lessons a dream or a nightmare, respectively. Figure 17 illustrates the main components derived from students' essays by grounded theory analysis. Because the original article [8] is published in Finnish, a summary of this sub-study is presented next, along with some examples of the data.

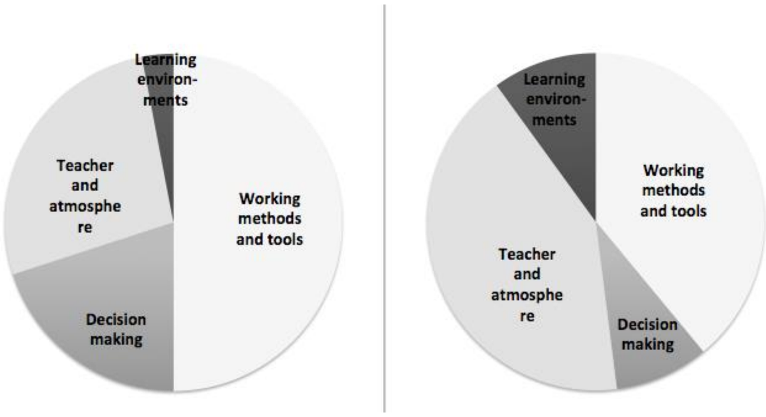

Figure 17. Main components in students' essays: Dream stories on the left and nightmare stories on the right (modified from [8]).

In both types of essays, working methods appear most often, especially in dream lessons. These related to interactivity, co-operation, manipulatives, homework, exams, and timing. In dream lessons, learning takes place in small groups of friends, proceeding by solving problems without any time constraints, and by developing games. Writing involves making only short summaries, and homework is only for revision. In a nightmare lesson, by contrast, learning is based on too many time-pressured tasks, and on copying text from the blackboard, as ordered by the teacher. Frequent tests are also a feature of nightmare lessons. At the end of this kind of lesson, the teacher would give a lot of homework related to basic tasks [8]

> *A dream mathematics lesson would be one in which everything must be carried out in pairs, and the lesson would not be full of tedious tasks, but we could play math games and design new games.* (Boy #1)
>
> *There would be no rush from anywhere to nowhere. There might be a few rather difficult task instead of piles of easy tasks.* (Boy #2)
>
> *During a dream lesson, you don't have to do notebook tasks or copying from the blackboard. Such lessons will easily get dull and my attention can be easily drifted away, and afterwards I recognize that I haven't learned a thing.* (Girl #2)

Opinions on decision-making also appear more often in dream lessons than in nightmare lessons. During dream lessons, students have freedom to choose where they sit, with whom they work, how much they want to work, at what tempo they work, and the content to be learned. During a nightmare lesson, students have no scope to influence the ongoing process.

> *The lesson would be nice or enjoyable, we could decide the groups and get our own friends around us, and we could talk about mathematics tasks if we want to, but we would not be forced to do so. So to say we are authorized to determine how much mathematics we want to learn*. (Boy #4)
>
> *One can choose from where to start with the tasks. For example, if the objective is to do tasks 1-20 on page 86, one could start from a more challenging task 14 instead of the easiest task 1*. (Girl #2)
>
> *The most terrible mathematics lesson would be the one where you should not sit with your friends. The teacher would be really nasty and harsh and she would give terribly lot of homework*. (Boy #5)

The atmosphere and the teacher become a nightmare where the teacher controls everything and treats students in a way that is unfair and biased. During a dream lesson, on the other hand, the atmosphere is fun and relaxed, and students can work in peace. The teacher is fair, nice, kind, cheerful, and inspiring. Students also mention a good atmosphere and the fact that students really learn something during the dream lesson.

> *Atmosphere at a mathematics lesson should be calm and focused. All of the students should try their best, and the atmosphere should be relaxed.* (Boy #6)
>
> *Our freedom would be limited to the minimum. We could not even chat with a friend, not to mention to go to the toilet*. (Boy #7)
>
> *Teacher does not judge anyone, even though they would not immediately master the topic*. (Girl #3).

The term "learning environment" is used here to mean only the physical environment (more properly described as "learning ergonomics"). This component plays a more prominent role in the nightmare stories, relating to such things as, for example, messy and ugly classrooms with uncomfortable chairs, poor ventilation, and a lot of echo.

When learning takes place in or between teams, enjoyment is increased because of the social nature of learning, which may explain the influence of the informal school in the learning process. Looking back the findings of the sub-study [8], students' expectations were in general aligned with the student-centered learning paradigm. These findings are supported by Hamre and Pianta's [83]

CEC research. Similarly, students' nightmare mathematics lessons equate with classrooms with a negative emotional climate (i.e., low CEC). According to Hamre and Pianta [83], these are classrooms where teachers and students share little emotional connection and regularly disregard, disrespect, taunt, humiliate, threaten, or even physically lash out at one another. Teachers in such classrooms do not design or apply lessons with students' perspectives or cognitive capabilities in mind, nor do these teachers diverge from a lesson plan when students' boredom, discomfort, or confusion becomes apparent. Phase 2 of the ClassPad-project shows that, although the students were worried at the outset about not having a teacher controlling their learning processes, the majority of the students (n = 21) realized by the end that self-guidance facilitated rather than inhibited their doing and learning mathematics. In addition, during project lessons the informal school culture came to greater prominence [92], and the significant improvement in students' cognitive skills indicates that CEC improved during the ClassPad project by comparison with students' previous experiences of mathematics lessons (cf. [84]).

5. Challenges for teacher education

5.1 Topic realization

Recalling the discussion in earlier chapters, the implementation of SCLE is one of the key issues in teacher education. The literature review reveals the difficulty of shifting teachers to a student-centered approach. Teachers' beliefs about mathematics, learning, and teaching are strongly reflected in how they teach. Reflection is assumed to play a key role in change of practice, and many researchers see a cyclical relationship between changing beliefs and changing practices (see [94], [95], [96]). Swan's [97] study revealed that, although teachers had differing beliefs about mathematics and its teaching and learning, they continued to use predominantly teacher-centered practices in their classrooms.

Regarding the matrix in Table 2, Silfverberg and Haapasalo [19] conclude that teachers have difficulty in changing their stereotypic views on teaching and learning, and even more difficulty in adopting progressive technology:

> *...the conceptual space constructed by cross-tabulating the various paradigmatic shifts can be used both as a theoretical tool in the political decision-making and educational planning and as an empirical tool by which we can study different conceptions of the present and the desired future realizations of the culture of teaching math in schools. Many of the prospective teachers of mathematics believed that the paradigm shifts actually change the culture of teaching mathematics in the long run. However, much must happen before the changes are possible. The curricula must be renewed, the paradigm shifts must be taken into account in textbooks, the teachers' professional competence has to be raised and teachers ought to be inspired to take responsibility for carrying out the renewing process.* p.737)

In respect of teacher education, perhaps the most optimal approach would be to include use of technology in the student teachers' pedagogical studies. Eskelinen [98] suggests that design of a technology-based learning environment, within an adequate constructivist theory and linked to knowledge structure, seems to be a promising way of helping students to understand the basic components of teaching and learning. In applying an educational approach that stresses the importance of conceptual knowledge, an educator needs to be sensitive to cognitive and emotional variables in the learning process. These findings contradict the view that computer skills for trainee teachers should be taught separately from information structures and pedagogical thought.

5.2 Topic reconsideration

In developing research-based theories for instructional praxis, Haapasalo [99] emphasizes the following tensions: (1) objectivism vs. radical constructivism; (2) developmental approach emphasizing procedural knowledge vs. educational approach stressing conceptual knowledge; (3)

Gagne's guided learning vs. minimalist instruction emphasizing student's volition to learn; (4) instrumentation, where technology is shaping the actions of doing mathematics vs. instrumentalization, where technology is also shaping the mathematical objects; (5) learning by instructional materials vs. learning by design; (6) teaching mathematical content vs. emphasizing sustainable heuristics from the history of mathematics; and (7) looking at internal problems of mathematics education vs. applying business principles to overcome the bad reputation of mathematics. He stresses the importance of responding to these seven challenges as follows, to develop solid, practical theories for teaching and teacher education:

- *to promote collaborative social constructions*
- *to link conceptual and procedural knowledge*
- *to resolve the conflict between a systematic approach and minimalist instruction*
- *to relate instructional design and assessment to instrumental genesis*
- *to promote learning by design*
- *to revitalize sustainable heuristics in human history*
- *to apply business principles to overcome the bad reputation of mathematics.*

The following three examples aim to show how these challenges can be met.

Prospective teachers' readiness to design SCLE

To see how the matrix in Table 2 emerges at an operational level when prospective elementary teachers (PET) plan their own mathematics lessons, two separate groups of trainees completing the course 'Mathematics Pedagogy' were studied in March 2009 and 2010 (N = 101 and N = 87, respectively). The subjects were given a learning material package for percentages, tailored to fit the MODEM framework (see Chapter 3.2). In addition, the material also contained conventional textbook tasks for which the research designers found only Reinforcement of percentage-calculations as a pedagogical function (see the right-hand side of Figure 5, p.12). The subjects' task was to make a plan and then to explain how to use this material on the primary level in an appropriate way. The research objective was to find out which paradigms of teaching and learning would emerge in the plans.

Each individual plan was analyzed according to the matrix in Table 2. The subjects also had to complete a self-evaluation, characterizing their own plan on the same matrix. In the 2010 study, the subjects also formed small teams to conduct the same kind of analysis of plans made by some other (anonymous) subjects. They also discussed the main differences between behaviourist and constructivist plans. From these essays, the ability to create student-centered learning instructions was studied through content analysis (chapter 2.2), and in particular by theory-driven content analysis [48].

Shifts in students' mathematical profiles

Following our findings from the ClassPad project (see Chapter 2.1), we used the same instrument for two groups who had recently commenced their university studies. The first group comprised PET undertaking their educational studies at our faculty. The other group had just started to study mathematics at the Mathematics Faculty and was not connected with educational studies. None of the students belonged to both categories. The study was designed and administered during the study year 2005–2006, when the author was teaching pedagogical studies to the first target group.

We aimed to discover (1) how the profiles of the two groups differed from each other, and (2) how the profiles changed during the first year of study. To measure the profiles, an identical web-based questionnaire was used in both September 2005 and April 2006. The first questionnaire was completed by PET (N = 116), and mathematics students outside the teacher education (N = 66). The sample sizes at the second stage were 22 and 50, respectively. For each of the three profiles described in Chapter 2.2, and for each of the eight Z-activities, questions used a Likert-type scale (see [7]). For each question, the student also had to express (on the same Likert scale) how sure he/she was about his/her opinion. From the student's answer, and the associated degree of certainty, a new weighted variable was defined that could be used to span student profiles. Because of the asymmetric and discontinuous data, the Mann-Whitney Test and Sign test (2-tailed) were used for statistical analysis. To ascertain how the students understood the questions, sub-samples from both groups were also interviewed.

Organizing progressive teacher education

To give prospective mathematics teachers (PMT) an opportunity to opt for authentic learning situations within the above-mentioned seven challenges, we reorganized part of their pedagogical studies as empirical action research-based inquiries. The examples demonstrate how the pedagogical studies of PMT and PET can be successfully integrated.

During the Classpad-project (2005-2006), one PMT research team [p1] used the MODEM framework to plan learning and assessment materials for basic concepts of statistics (e.g., organizing and representing data, mean, standard deviation, mode, and median). Another PMT team [p2] did the same in respect of the basic features of linear functions. A third PMT team [p3] measured pupils' knowledge of these concepts using the same test as in the MODEM 1 study (cf. [60]). In designing these learning environments for the learning of linear functions and statistics, the teams combined features of the systematic approach with minimalism. PMT Karttunen et al. [p4] observed and videotaped pupils while learning, and analyzed their thinking processes when studying linear functions. Two further PMT research teams investigated student profiles. The first one [p5] measured the profiles of PET and first year mathematics students. The second PMT team [p6] analyzed the profiles of 9th grade students.

In the later studies conducted in 2009, one PMT team [p7] conducted a microteaching session with PET (N = 82), in which their control group (N = 40) tried to find a viable definition for the vector concept within collaborative socio-constructivist teamwork. Both this group and the comparison group (N = 42) underwent a test concerning basic features of constructivist learning. In the 2010 PMT Study, the same kind of microteaching was conducted for all subjects, with no control group [p8]. PMT Hiltunen et al. [p9] prepared video clips of microteaching lessons and used them for stimulated recall: Subjects discussed in teams how to analyze the purpose of each pedagogical measure as it appeared in the current clip, on which framework theory it was based, and those measures were experienced by those subjects in the role of pupils. The PMT teams of Rissanen et al. [p10] and [p11] analysed lesson plans among PET as described in the first sub-section. PMT Kekki & Keronen's team [p12] analysed 7th grade students' (n = 17) dream and nightmare versions of mathematics lessons. This analysis was conducted using the same data as described in the final sub-section of 4.3.

5.3 Empirical evidence

Prospective teachers' readiness to design SCLE

Eronen and Haapasalo [7] summarize the findings from these studies. The most important of these conflicts with the general view expressed by PET in the survey study after their pedagogical lessons. The findings indicate that PET had major difficulty in even identifying the pedagogical function of each task type in the ready-made learning materials, and that they tended to use a teacher-centred approach. The analysis from the 2009 and 2010 studies (see [p10], [p11]) similarly confirms that the subjects tended to use the materials within a teacher-centred approach. In all cases, this meant that collaborative knowledge construction inside student teams or between the teams was generally neglected. Table 3 shows the plans quantified and categorized. In the 2010 study, to get a more dynamic view of student plans, this classification was done according to how many indicators (or so-called "visits") could be found for each category in each plan. This analysis revealed interesting paths between the cells: some subjects travelled through cells, while others stayed mainly in the lower left corner, making just brief visits to other cells (Table 3).

The self-evaluations among PET showed that those who described their plans as behaviorist recognized their background theory quite well, describing the necessity of teacher-centered leading in the standard way. Almost all of these subjects wanted to start teaching by using the same tasks, copied from the mathematics textbook. Our analysis revealed that the subjects overestimated the constructivist spirit of their plans, especially with respect to socio-constructivism.

Table 3. The lesson plans among PET, categorized by researchers in 2010 study according to how many visits to each cell were recognized (modified from [p11]).

(N=87)	NON-COOPERATIVE	COLLABORATION INSIDE A TEAM	COLLABORATION BETWEEN TEAMS
MINIMALIST INSTRUCTION	2	6	0
STUDENT-ORIENTED	20	36	4
TEACHER-CENTERED	236	130	10

Shifts in students' mathematical profiles

It is worth focusing here on the most interesting findings from Haapasalo and Eronen [7]. At the beginning, all three profiles were similar among PET and mathematics students, emphasising calculation and argumentation. Subjects thought that a computer mainly supports calculation but argumentation very weakly. As regards changes during the first study year, there were no significant shifts among mathematics students in any of the Z-activities, for either Math-profiles or Techno-profiles. In addition, the only significant shift in the Identity-profile appeared in *calculating*—which, surprisingly, showed a tendency to increase. These findings can be interpreted as suggestive of a very old-fashioned ”paper media culture” in the department of mathematics, stressing pure mathematics without any great interest, for example, in practical applications or the making of one's own hypotheses. This even manifests itself through the students' expressions in everyday language, when they refer to ”calculating exercises”, even in questions about a course in abstract topology.

Figure 18 illustrates profile shifts among PET. A significant shift in Math–profile appeared for *playing* (an increase) and *calculating* (a decrease), the latter being the only decreasing dimension. At the end, subjects emphasized *application, argumentation,* and *playing*, whereas *evaluation* appeared weakly. They were more confident of their answers than at the beginning, and significant growth occurred in *finding, construction,* and *evaluation*.

By the end, the PET showed more rounded Math-profiles and Identity-profiles than the mathematics students, especially in relation to the creative components on the right-hand side of the octagon. A significant difference appears in *playing, construction, application and calculation* (the latter appearing to be stronger, as mentioned above). In the Techno-profile, the above trend appears in *finding, construction, argumentation, playing,* and *ordering*. For the first three, the difference is significant.

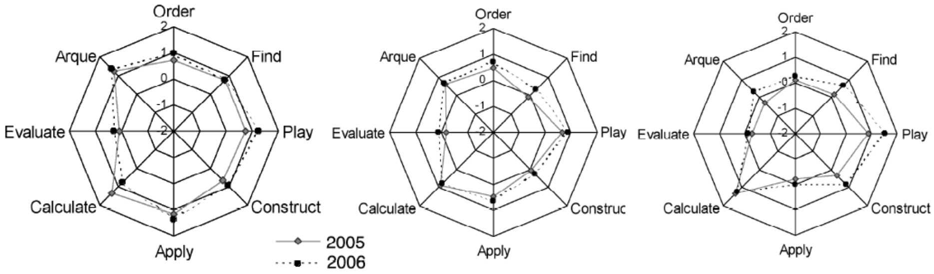

Figure 18. Profile shifts among PET: Math-profile on the left, Identity-profile in the middle, and Techno-profile on the right. [7]

Organizing progressive teacher education

The relatively degenerated profiles among mathematics students suggest that conventional mathematics education (in the sense of mathematics teaching) (cf. [27], [28]) seems to be counterproductive in relation to the extension of students' views of mathematics. However, our progressive pedagogical solutions for PET, promoting creative activities within a constructivist framework and using modern technology, seems not only to change students' views of mathematics but also to increase their self-confidence in using both mathematics and technology. We did not choose a progressive way to integrate students' pedagogical studies because it is easy, but rather because it is hard. Having been accustomed to ready-planned materials and objectivist-behaviorist teaching during their own learning history, the demand for continuous improvisation seemed to cause uncertainty, especially at the beginning of the project. However, by the end, PMT expressed mainly positive opinions in relation to this effort, which they described as progressive, challenging, and worth the risk. With respect to the PMT-objects' own research concerns, it was most important to them that they gained experience in conducting relevant, valid, original, and rigorous educational research, in which mathematics is more than just a place holder (cf. [2]). And despite the gaps in their data analysis and interpretation of results, we also learned a lot as supervisors. The project yielded a number of new research ideas and, with the students' kind permission, we may also use their data and pre-analysis to improve our methods.

6. Looking back and looking forward

With particular reference to mathematics teaching, student-centeredness has commonly been seen more or less as Swan [97] suggests: that in student-centered learning environments,

> *the teacher takes students' needs into account when deciding what to teach, treats students as individuals rather than a homogeneous body, is selective and flexible about what is covered and allows students to make decisions, compare different approaches and create their own methods. Instead when using teacher-centred practices, the teacher directs the work, pre-digests and organizes the material, gives clearly prescribed instructions, teaches everyone at once in a predetermined manner and emphasizes practice for fluency over discussion for meaning*. (p.62)

This thesis takes a position against these commonly-held views that the SCLE contexts *in* and *with* (see Table 2) might be less useful where there is some specific content to be learned. This position may well be a decisive factor in the failure to achieve positive shifts in instructional paradigms, as referred to by Lea et. al [73] in relation to the differences between educators' acts and thoughts. It seems that to highlight the contexts of *in* and *with*, it becomes necessary to ask whether mathematics can be learned without a common specific content, and whether emphasis should instead be given to the processes of doing. The Classpad-project has suggested new ways of organizing versatile SCLE, in which instead of giving students a mathematical content to be learned, students are asked to clarify how ClassPad creates algebra from geometrical representations, and vice versa. In this way, we could offer students a starting point for learning that gives more emphasis to 'in' and 'with'; and we could organize a multidimensional learning enviroment in which student processes occupied the upper row of the matrix. Student teams could choose—and, indeed, create—any object they wished. In this way, mathematical problems were seen to become important when they were psychologically meaningful for the students, as Haapasalo and Samuels [44] discuss in relation to educational robotics. This may be one reason behind the positive development in students' Math–profiles and Identity–profiles. The study shows that autonomy and the opportunity to collaborate improves emotional climate, and also explains students' improved cognitive development (cf.[84], [100], [101], [102], [103]). The influence of informal school culture therefore becomes apparent in SCLE. The fact that students' Techno–profiles changed, as in the work of Samuels and Haapasalo [38] on learning technologies, is discussed in Chapter 2.2.

The implications here must be considered with regard to the whole school culture. As the role of informal learning increases, not least because of progressive instrumental genesis, the learning focus could usefully be shifted from the classroom to students' free time activities:

> *which can stimulate modelling processes, for which school could take, referring to a car race, the role of a pit stop (to orchestrate technology-based investigation spaces which allow students to explore spontaneously the facility of real and virtual environments which are both, meaningful to them and their community, and which naturally motivate a greater use of mathematical language in its different forms [104]*.

Our ongoing efforts to implement this pit stop culture in school and teacher education seem to be promising but difficult. A study of upper secondary school students by Haapasalo and Vilpponen [105] suggests that students use ICT for entertainment rather than for those purposes for which computers were originally invented. Based on student feedback, they suggest that positive outcomes could be achieved by the pit stop philosophy if it were applied over a longer period of time. Students' hesitation was seen to be linked to technical questions rather than to any general mistrust of the pit stop approach itself, and their descriptions of dream lessons contained many attributes of the pit stop: freedom to choose with whom and where to work, at which tempo to work, and opportunities to choose content that is more connected to playfulness (cf. [42]).

As regards teacher education, all the researches we conducted among PET reveal poor mathematical skills in respect of even the most basic level of elementary school mathematics. The constructivist paradigm makes new demands on teachers concerning knowledge structure, and it is therefore teachers perhaps easier for them to choose a method that is familiar from their own learning history and conforms to the types of task that appear in textbooks On this basis, they continue to place their trust in behaviorist theories and objectivist knowledge, as described in Haapasalo and Eronen ([7],[32]).

At the first stage of their follow-up study among PET, Haapasalo and Eskelinen [33] found that these subjects also seem to use ICT for entertainment purposes rather than for work requiring or promoting Z-activities. However, they found that utilizing ICT as learning technologies in the constructivist spirit during pedagogical studies, as Samuels and Haapasalo [38] suggest, altered both Math–profiles and Techno–profiles. Reference is also made to a remarkable shift in paradigm of teaching and learning (cf. also [32]). The fact that PMT were able to conduct quite demanding research in mathematics education within our ClassPad project shows that even ambiguous goals can be attained using appropriate pedagogical solutions. All of the PMT –students involving in the sub-studies as researhers were highly motivated during the whole working period, and their work can be characterized as professional. In their plans, they showed constructivist features and flexibility, and their own evaluation of their plans was closer to that of the researchers. Their enhanced mathematical skills cannot account for this because improving those skills alone does not necessarily improve profiles in any relevant way (see [7]).

With regard to the models developed through grounded theory, our ongoing iterations of the ClassPad project focus on testing those models deductively in varied working environments, as well in schools and in teacher education. Given that the main challenge in teacher education may be to improve synchronization and integration of course modules, and to orchestrate authentic technology-based SCLE experiences instead of lectures, perhaps the most urgent problem in the teaching of school mathematics is assessment. We are working to find prototypes for new kinds of problems to be used for assessment, independently of whether or not they are solved by use of technology. These ongoing efforts are conducted in the spirit of Haapasalo [37], in seeking to explain why conventional task types lead to a dead end for assessment where technology is used. I share his opinion that

> *most of students' instrumentation and instrumentalization often happens in his or her free time especially when using hands-on technology. Thus, educators should shift their focus from well-prepared classroom lessons to scaffolding students by guiding them in recognizing, interpreting and solving problems also outside the classroom. On the other hand, allowing the utilisation of technology in teaching and examinations does not necessarily improve students' understanding and motivation if technology is merely used for solving conventional task types.* (p.91)

By analyzing the Finnish matriculation examination in mathematics over the past ten years (see http://matta.hut.fi/matta/yoteht/), PMT, Vienonen, Kempas, and Hakkarainen [p13] found that most of the tasks, especially in advanced mathematics, can be solved with a CAS calculator, without any deeper understanding of the underlying mathematical concepts. Once they had guided my students in this way (at the so-called "ClassPad cafe") for a short period of time before the matriculation examination, these students were empowered to cope successfully with the tool. Although they thought the tasks for learning mathematics should be radically transformed to fit the use of technology, the students were still were quite hesitant as to whether the conventional task types should be replaced by the new task types for the purpose of assessment. These experiences again confirm the somewhat static beliefs among the different interest groups in respect of mathematics education.

7. Coda

To emphasize the main findings and the fact that technology appears as innate tool for today's learners, the term 'technology' does not appear in the title of the thesis. For the same reason, when discussing the professional development of prospective teachers, more or less populist paradigms as TCPK/TPACK, for example, have been dropped. A complex interplay of Content (CK), Pedagogy (PK), and Technology (TK) appers rather as an implication than a "tail wagging the dog", namely.

Regarding this article, in ideal socio-constructivist collaboration, the learning happens in negotiations between student teams and therefore the power regarding those determinants shifts from the individual students to the student teams. Therefore, the Neuman's [81] triple-step contexts should be extended by the paradigm *between* student teams and the term 'student-centered' should be redefined to minimize the role of the teacher as decision-maker with regard both to learning objectives and working methods.

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