Understanding Geometric Pattern and its Geometry (part 1)

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In this paper, we explore selected mathematical concepts of art with geometric patterns as its major component. Such art includes carving patterns in stone; various types of geometric mosaic; or assembled wooden structures, known as kundekari art, into a sophisticated decoration of doors, window shutters, or furniture. We discuss here examples of real patterns taken from medieval and modern architecture. We show their constructions and geometry hidden behind these patterns.

Introduction

In recent years there is growing interest in art with some geometric influence. In this paper, we will explore selected examples of segmented geometric patterns. This type of art in Central Asia is known as "gereh" (Iran), or "girih" (Uzbekistan). Western authors refer to it as Islamic geometric patterns, although there is nothing Islamic in these patterns. In this paper, while talking about geometric patterns we will use the Persian name "gereh" or just "geometric pattern".

In all Muslim, and nearby countries, geometric patterns are considered as a form of traditional art, and styles of these patterns differ from a place to a place. Even in the same region, geometric patterns may be different for different epochs. For example, in Turkey, geometric patterns from the Seljuk era are different than geometric patterns from the Ottoman period. There can be also serious differences depending on the material used for rendering a gereh. For example, marble or other stone used as a material for rendering patterns forces artists to develop designs without very small elements. At the same time patterns rendered in the form of painting on ceramic tiles may have very sophisticated tiny shapes.

There are always questions – why a person is interested in pattern design, how much and what kind of geometry he, or she, needs to make a successful job? We can imagine a few scenarios. Let us mention three of them.

Scenario 1

An artist wishes to use geometric patterns in his artworks. Such a person is usually more interested in using colors or textures. His knowledge of geometry is usually very minimal and shapes in his patterns may not be as accurate as a mathematician would expect. He may also, from time to time, apply deliberately some distortions to his geometric artworks.

Scenario 2

A mathematician while designing geometric patterns may use precise geometric constructions executed with compasses and ruler. He may also investigate shapes included in geometric patterns, proportions, and relations between particular elements of a pattern, symmetries global and local. In this case, we deal with classical constructive geometry and the tessellation theory.

Scenario 3

A craftsman rendering a pattern in any concrete material will be even more in need of highly accurate geometric constructions, uniform shapes, and relations between sizes of these shapes and the size of the complete pattern. Patterns were always created for a precisely defined space. Thus he has to calculate very accurately the size of each element of his pattern. A small mistake in his calculations may lead to a disaster – a final design may not fit onto the provided space, or it may be too small for it. To save his time he may also prefer to have designs with only a few different shapes that will be repeated in many places of his work.

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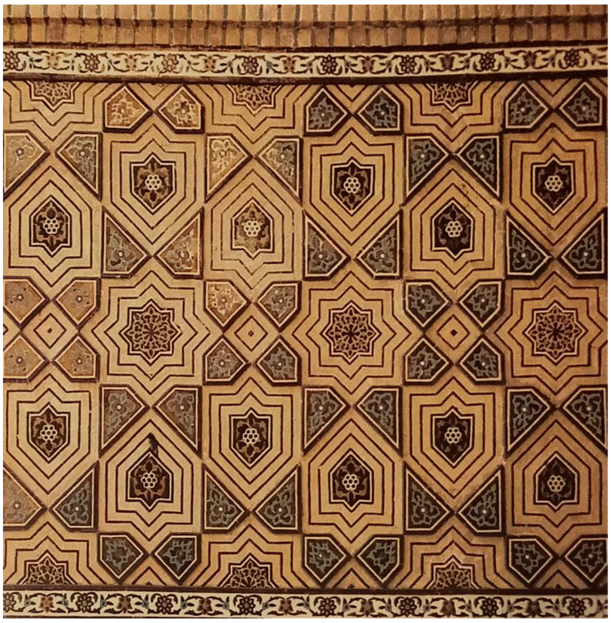
In this paper, we will be following the second approach, i.e. what a mathematician, or a teacher of mathematics, can find in geometric patterns and how he can fit them into his teaching curriculum.

Example 1 – A Persian design using clay tiles

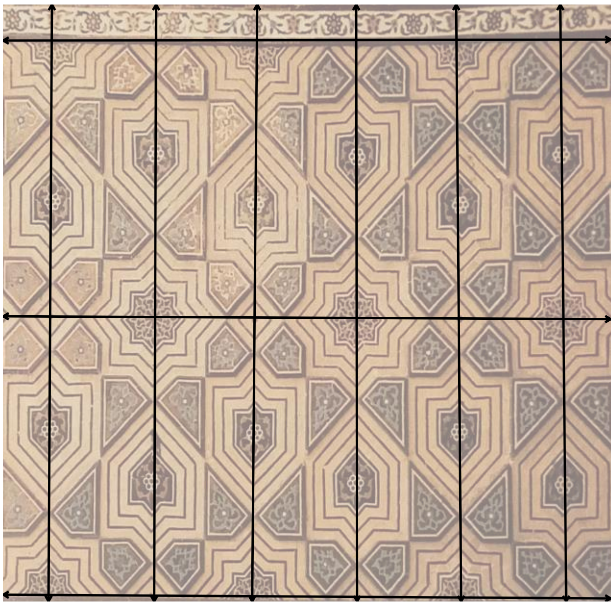


The Analysis of Persian design

The photograph shows a gereh that usually is made using clay tiles with some extra decorations. We have here only 5 shapes – a star with 8 vertices, a square, a long kite, a pentagon, and a specific octagonal shape with four long edges and four short.

Each of the shapes used in this design has at least one symmetry line (a mirror). Two of these shapes have more than one symmetry line (square – 4 symmetry lines, star – 8 symmetry lines passing through its center).

To create artwork using this design we will need a given number of ceramic tiles of each of the mentioned shapes. We have to know how many such tiles of each type we will need and what should be their size.

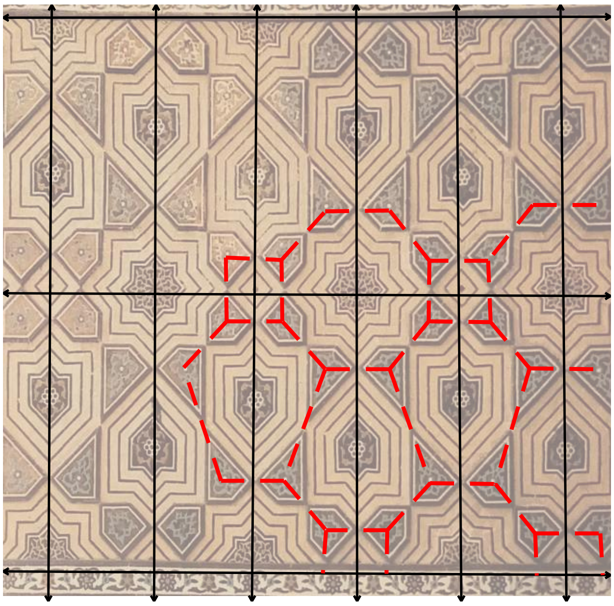


Designing considerations

One can easily notice that this pattern has some vertical as well as horizontal symmetry lines. Here they are marked using solid lines. Each of the areas between these lines contains the same geometric motif or its mirror reflection. Thus to construct this pattern we need to construct the content of one of these rectangles and then use multiple copies of it. This way we can easily plan how a given space can be covered with this pattern.

Such a rectangle in Central Asia is called rapport or module. In western literature, authors use the term repeat unit or a template.

Such templates were often used by medieval designers. Their scrolls contain collections of various rectangles filled with some geometric motifs.



A tessellation

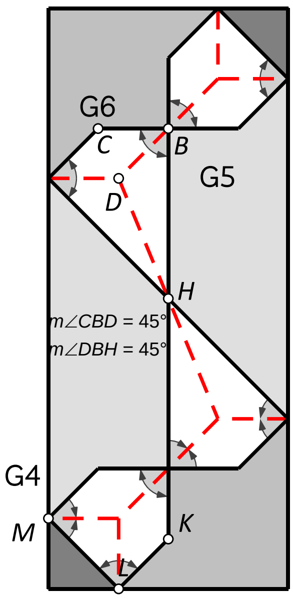
One can take a pencil and draw by hand polygons around some shapes in this pattern. These polygons form a tessellation of the plane. Each of these polygons has the same symmetry lines as the pattern on it. This tessellation forms the hidden geometry of the pattern. Most of the known geometric patterns were created or can be created, using tessellations. A tessellation for a geometric pattern has the following properties:

G1. Tessellation polygons are convex and symmetric.

G2. The tessellation is an edge-to-edge tessellation

G3. If a tessellation polygon is intersected by the edge of the contour (here the solid lines), then it is always intersected along its symmetry line.

As we will see later the same tessellation can be used to create a few different patterns and often the same geometric pattern can be created using two different tessellations. Let us look at the relation of a pattern with its tessellation.



The relation between a pattern and its tessellation

The drawing shows a template for our gereh. We have here a contour – the black frame around the whole drawing; tessellation tiles (dashed lines) and pattern (solid lines). Many known patterns follow the following rules:

G4. If a single line of the pattern touches the edge of a tessellation tile or edge of the contour, then on the other side of the edge, or of the contour, there is another line going into the mirrored direction. See the point M.

G5. If two lines of the pattern meet on the edge of a tile, or of the contour, then they continue without bending on the other side of the edge or contour. This simply means that the angles between lines of a pattern and edge of the tessellation tile are equal. See point B and many others.

G6. If two directions meet inside the tile then a pattern following them may bend in this place. See point C.

G7. A line of the pattern may end only on the edge of the entire pattern.

The rules **G1...G7** presented here we call the gereh rules. The majority of known patterns follow all these rules. However, there are patterns where some of these rules were broken or ignored partially. The famous design on the external gate of the Topkapi Palace is one of them. Another such pattern we can find in the Fatih Mosque in Istanbul. A good example of such gereh is a design from Bursa Green Mosque. A few other examples can be seen on one of the Bukhara scrolls.

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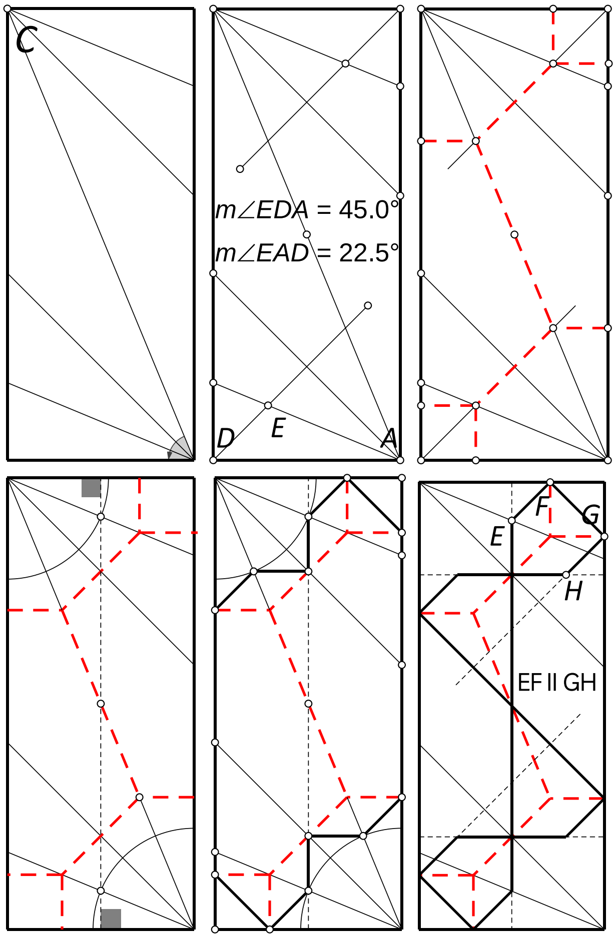
We will come back to the geometric properties of gereh and its tessellations later. Now, let us see how a pattern discussed here can be constructed.



Step-by-step construction of the pattern from example 1

This series of drawings shows all major steps in constructing the pattern. These steps include:

[First row] Construction of the tessellation

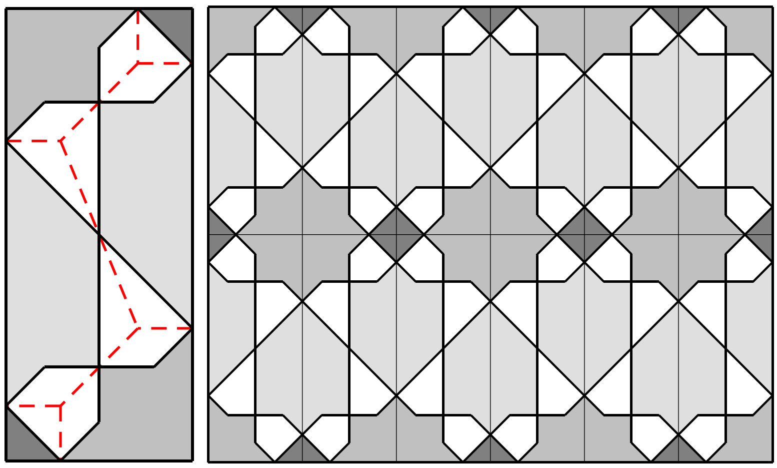
[Row two] Construction of the pattern

[Row three] The final template still with tessellation and a larger pattern created from this template using 6x2 copies of the template.

The contour was created by drawing a horizontal segment and two lines perpendicular to it. Then we divide the bottom right angle into 4 equal parts. The point of intersection of the third section line, counting from the bottom, with vertical line marks the top edge of the contour (the point C in the first drawing).

The vertical dashed line in the second row is the starting step to draw the star in the quarters of octagons and lines of pattern in other polygons.

Note – each time when we draw a pattern for any polygon we follow the rules G4...G7. This means we always care about having equal angles on both sides of the tessellation edges.



Complete template and a pattern

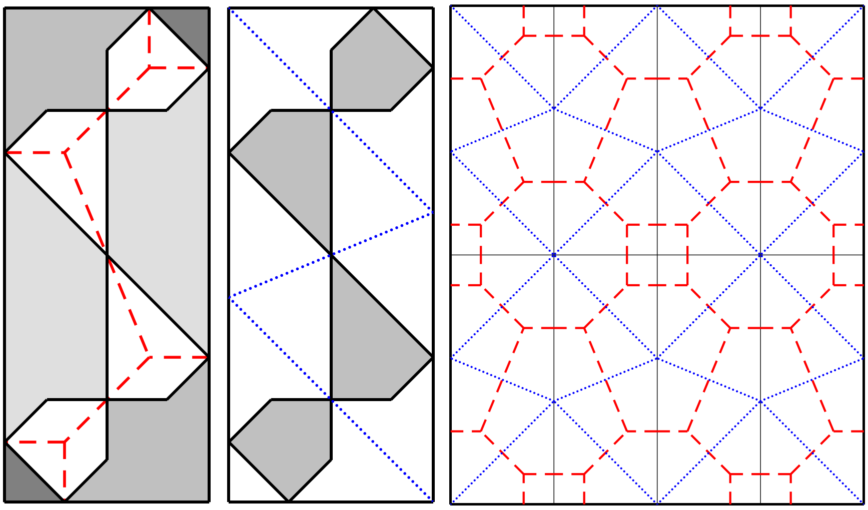
[Far left] The drawing shows a complete template with edges of tessellation tiles. In some of the old drawings of geometric patterns, the tessellation edges are often drawn in red color. For the final pattern, we have to remove them.

[Right] A real pattern made from this template may contain any number of copies of the template or its mirror reflections. The thin lines should be removed.

The method of design, presented here, in the modern industry is called top-down design. We start from an overall framework, then we build a structure and finally we construct all the details. Medieval designers to get a precise application of a gereh had to follow this scheme. In Central Asia, we can see massive walls covered perfectly with a pattern using an even number of repetitions of a template. In many countries, especially in Turkey, we can see huge doors with a geometric pattern made from four copies of the template.

Two different templates for the same pattern

As we said before many geometric patterns can be produced using two different templates. Example 1 is a good sample to see how this property looks.



Two tessellations for the pattern from example 1

On the left side, we have two different tessellations for our pattern. The right drawing shows the relation between these two tessellations.

It is worth noticing that each tessellation can be used to create the same pattern. The one with dotted lines is even simpler to construct and use.

This last drawing is an interesting example for a mathematician – the network of dotted and solid lines in modern geometry is known as a Delaunay triangulation, and the network of dashed lines is known as the Voronoi diagram for this triangulation. It is also interesting that both Voronoi diagrams and Delaunay triangulations are very popular tools in modern art. In modern geometry, these two tessellations are often called dual tessellations. From both points of view, artistic and mathematical, we got very interesting objects that can be investigated in the form of students' projects. There is always a question for which geometric patterns we can get two different tessellations, for which of them we can produce Delaunay triangulation? This is the area where we may find many interesting subjects to investigate.

On global and local symmetries of geometric patterns

In geometry, we often discuss the symmetries of objects. Usually, we mean by them global symmetries. The theory of symmetry groups discusses global symmetries of patterns. This means we investigate special points on the pattern where multiple symmetry lines cross or points which can be treated as invariant points for rotations about a certain angle. Unfortunately, the theory of symmetry groups is too rough to describe some specific features of geometric patterns. Thus we have to look into local symmetries.

Definition: We say that a convex area A of a pattern, or a tessellation, with at least two different points, has the Dn symmetry if it contains a point in which exactly n symmetry lines of A intersect.

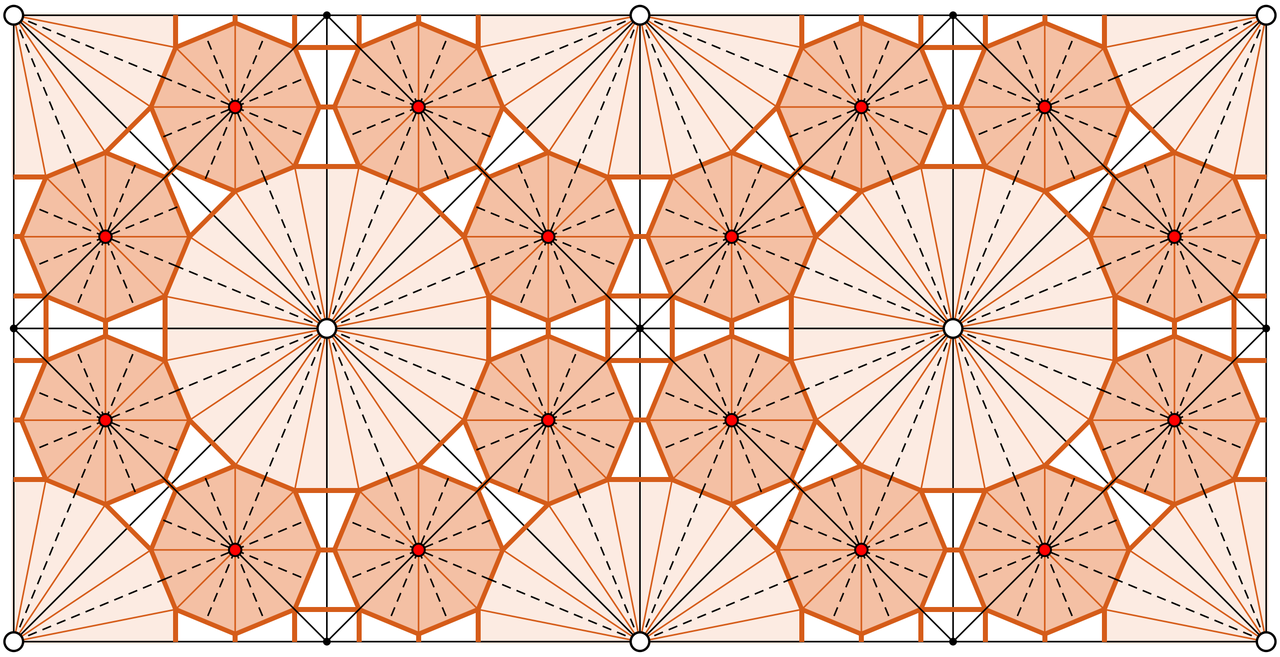
Definition: We say that a convex area A of a pattern, or a tessellation, with at least two different points, has the Cn symmetry if it contains a point invariant for rotations of A about angles $360/n$.

Definition: If a convex area has Dn or Cn symmetry and this symmetry cannot be applied to the whole pattern then we call it a local symmetry Dn or Cn respectively.

A geometric pattern may have global as well as local symmetries. These symmetries can be D type or C type. The expression $\{Dn_1, Dn_2, \dots Dn_k\}$ will denote a set of all symmetries for a given pattern or a tessellation, for example $\{D20, D10, D5\}$ or $\{C12, C6\}$.

Example 2 – A tessellation with hexadecagons, octagons, and squares

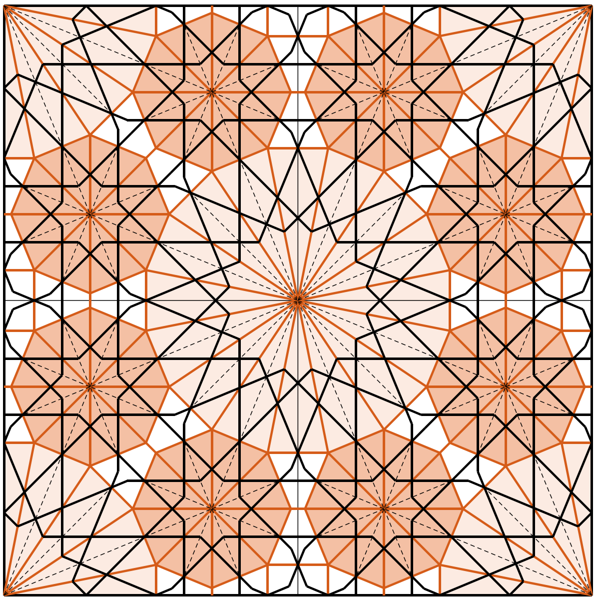
The next drawing shows a tessellation (thick lines) for a geometric pattern. The black thin lines are the mirrors for the whole tessellation. The dashed or color lines are the mirrors for selected areas only. Thus we can distinguish here octagons that are areas with $D8$ local symmetries and hexadecagons with $D16$ local symmetries. Note, a limited area around each hexadecagon can be considered as $D8$ local symmetry, but for the whole tessellation centers of hexadecagons are centers of only $D4$ global symmetry. We can find also areas outside of hexadecagons where we have $D4$ and $D2$ symmetries.



Map of global and local symmetries for an hexadecagonal pattern

Suppose that the shown here tessellation extends up and down as well as left and right covering the whole plane. The large points show places where 16 local symmetry lines cross, and spaces inside hexadecagons are the areas for local D16 symmetries. The medium size points show places where 8 local symmetry lines cross. The octagons are areas where we have D8 local symmetries. Small black points show places where 4 symmetry lines cross. These lines are global symmetry lines for the whole tessellation.

A tessellation shown in the above drawing can be used to create a geometric pattern with local symmetries $\{D16, D8, D4, D2\}$ or local symmetries $\{C16, C8, C4, C2\}$. In this last case, the points in the drawing turn into points of local or global rotations and there will be no mirrors.



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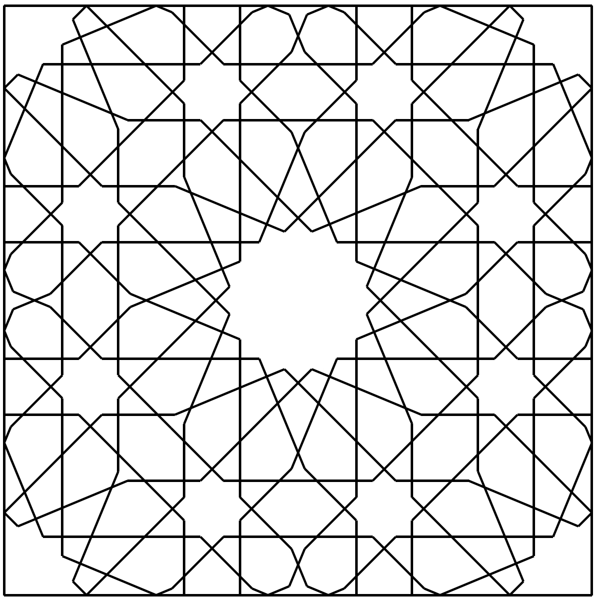

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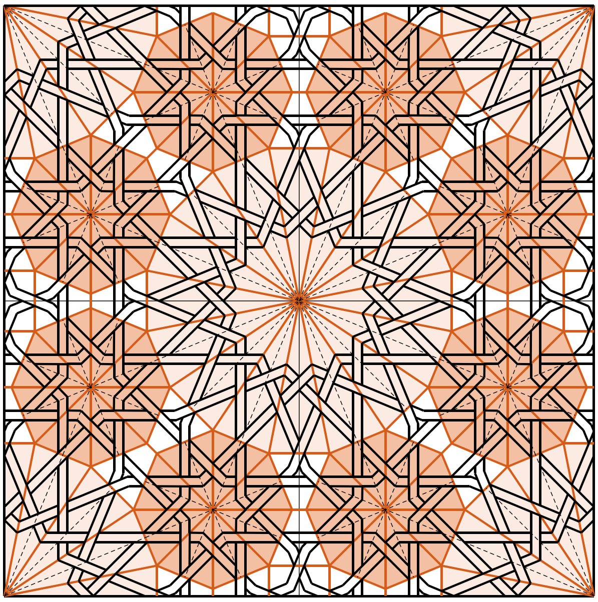

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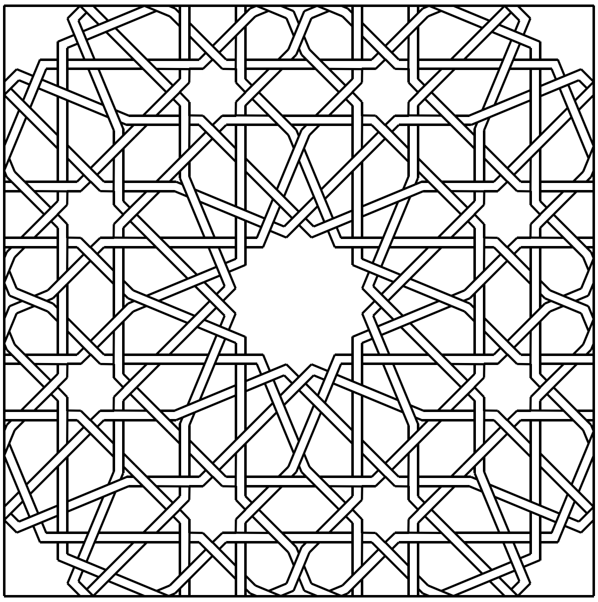


(4)

Various features of global and local symmetries

(1) In this drawing, we show a pattern created on our tessellation. The pattern follows all symmetries of the tessellation. Here we can easily see which areas have local or global symmetries $\{D12, D8, D4, D2\}$.

(2) Here we have only pattern and no tessellation. It is still easy to distinguish all local and global symmetries D16, D8, D4, and D2.

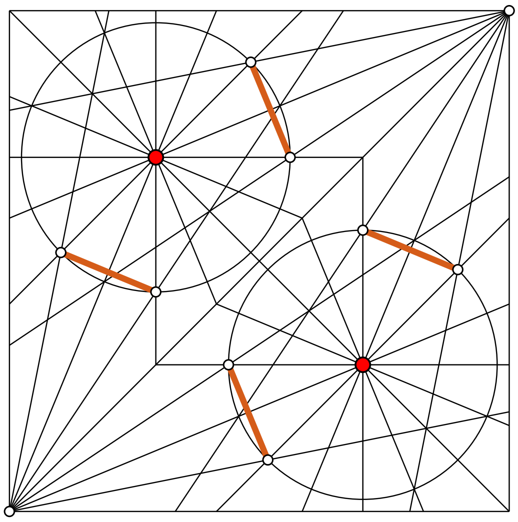
(3) In this drawing, we have tessellation with the interlaced pattern. Although in the tessellation we have several symmetry lines (mirrors) – global or local, in the pattern we do not have mirrors. Instead of mirrors and D-type symmetries, we have areas with rotations about 22.5, 45, and 90-degree angles.

(4) The last drawing shows the same interlaced pattern without tessellation and areas marking various local symmetries. Here we can easily see why we do not have mirrors in this pattern but we have there only rotations.

The difference between the two drawings (2) and (4) goes even further. Design (2) can be used to cover the whole plane by taking reflections of it about its edges. Design (2) cannot be reflected through any line. If we want to create a larger design from it then we have to use translations only and the vectors of these translations should be parallel to the edges of the contour and have lengths equal to the length of the edge of the contour.

Local symmetries for a geometric pattern can be compatible or not. Examples of compatible local symmetries are $\{D16, D8, D4\}$ or $\{D20, D10, D5\}$ or $\{D12, D6, D3\}$, etc. Here each index is divisible by any smaller index. The largest symmetry index is used to express the type of pattern. Thus we can have hexadecagonal patterns, or octagonal patterns (only symmetries $\{D8, D4\}$), decagonal patterns $\{D10, D5\}$, etc. In case when some divisions cannot be performed we talk about weak compatibility. An example of local symmetries with weak compatibility is the sequence $\{D12, D9, D6\}$. We can also find patterns combining stars or other shapes with an incompatible number of elements, e.g. stars with 11 and 9 arms, or stars with 13, 10, and 8 arms. In such a case, we usually do not have local symmetries $D11$ and $D9$, or $D13$, $D10$, and $D8$. These shapes often have some deformations that are not easily seen without proper measuring. Patterns with compatible symmetries are usually relatively easy to construct.

Before we will proceed further it could be worth looking at the design of the tessellation of the discussed here pattern. As for many other tessellations we have to create several regular polygons with common edges. Thus by constructing the common edges, we get outlines of the two octagons and quarters of two hexadecagons.

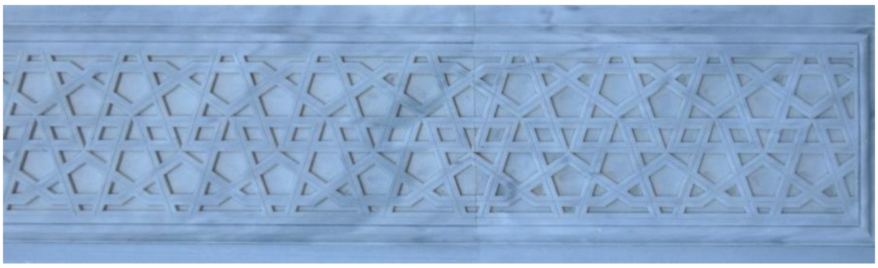


Construction of the tessellation for example 2

This drawing demonstrates the idea of creating tangent octagons and hexadecagons.

Start from a square contour. By dividing the two opposite right angles of it into 8 equal parts we obtain the two red points (dark in b&w printing). Then we divide the angle 360 degrees around each of these two points into 16 equal parts. The thick segments joining crossings of these lines are the common edges of octagons and hexadecagons.

This principle is useful in many other designs of geometric patterns. For example – for the large number of decagonal patterns from Iran we need to create the first decagon and then all other shapes are created using copies of the first decagon.



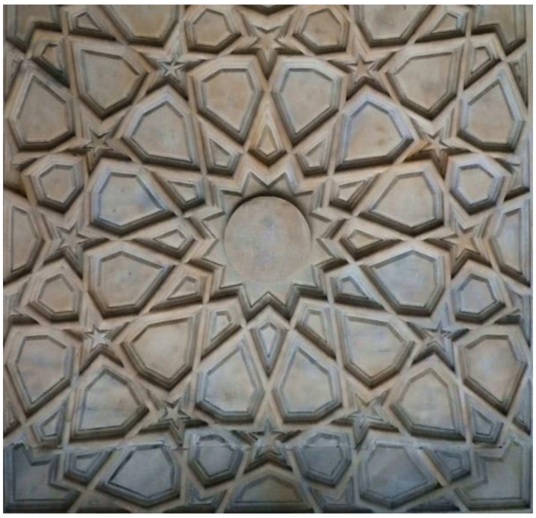
Example of an unusual decagonal pattern

Presented here pattern, despite not having stars or rosettes, uses shapes derived from the geometry of a regular decagon.

There are many designs where are no stars and no rosettes but we still call them decagonal, or octagonal, etc. patterns. This depends on the angles and shapes of tiles in the tessellation for such patterns. Such patterns have often one important feature. They are easily expandable in two or four directions. The pattern from the photograph can be expanded easily to the left and right as well as up and down.

Stars and rosettes

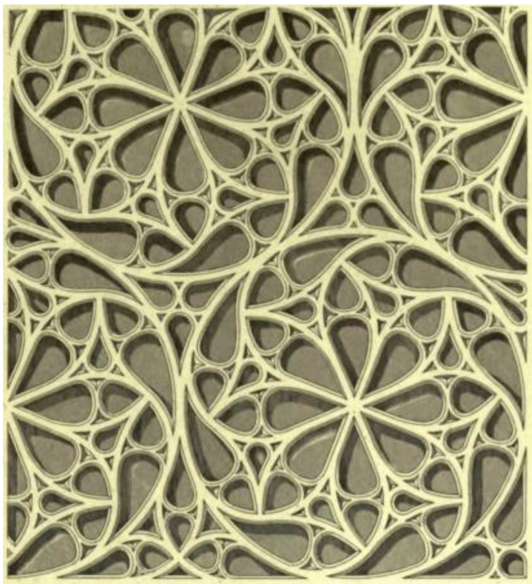
A very specific feature of geometric patterns from Central Asia and other Muslim countries are rosettes. We do not have them in patterns from other regions. However, in Gothic tracery, we can see some similar constructions.



A typical rosette from the Middle East

This photograph was made in Isfahan, Iran. One can easily notice that this is a dodecagonal rosette. The central star has 12 corner vertices and the rosette has 12 identical petals. Thus it has D12 local symmetry. We have here also quarters of identical rosettes in each corner.

Throughout Central Asia, the Middle East, Minor Asia, and Northern Africa, we can find similar rosettes – octagonal, decagonal, dodecagonal, etc. Most of them can be constructed using the same or similar techniques. While constructing such a rosette we may have various goals. One of them is to make starts between petals of rosettes as regular as possible. Another goal can be to produce rosettes with petals having parallel edges.



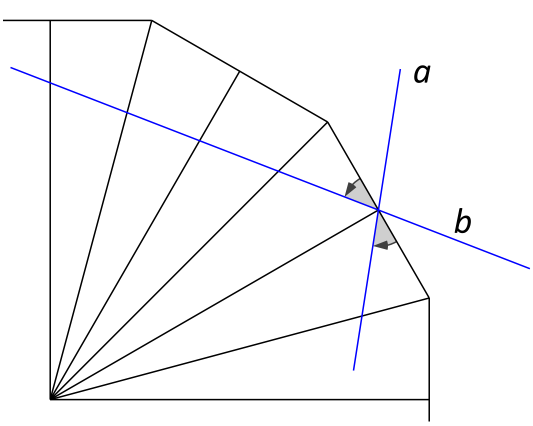
An example of a rosette from the Gothic tracery

This drawing was copied from the book "The Geometric Tracery of the Brancepeth Church, IX The County of Durham, Illustrated by Robert William Billings, London 1845". We can see here the major differences and similarities between geometric patterns from Muslim countries and Gothic geometric patterns, so-called Gothic tracery. In the first group, we have mostly segmented designs and usually with symmetries of the D type. An exception are interlaced patterns. While in Gothic tracery we have mostly curvilinear designs based on tangent circles and arcs. Many of these designs have local symmetries of the C type.

We can easily notice that here the motif inside of each large circle has local symmetry C8. Motifs between large circles have local symmetries D3.

The star and rosette shapes can be constructed in a few ways. In example 2 we demonstrated the construction of a star. The construction of a rosette is more complex. However, still, we can use an appropriate regular polygon.

A typical situation is when we need to construct a quarter of a rosette in two opposite corners of the contour. Then by dividing the opposite angles into a given number of equal parts, we get a framework for the rosette construction.



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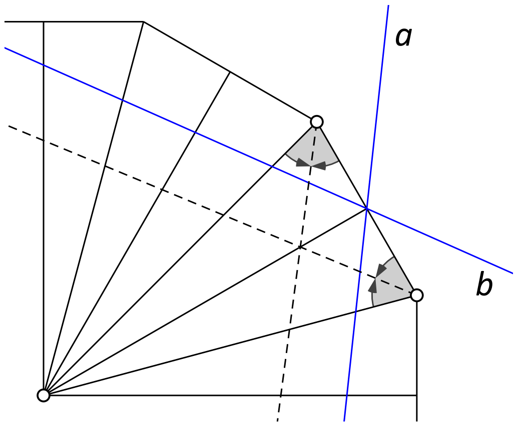

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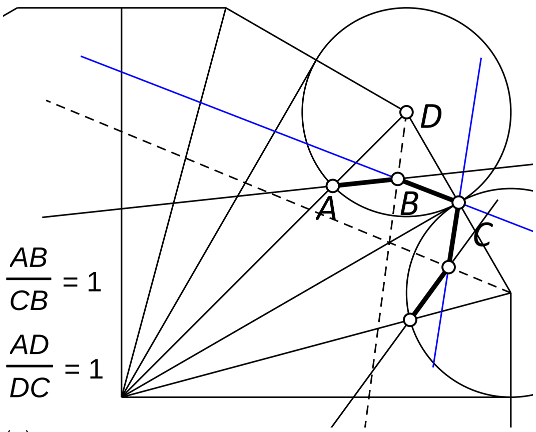
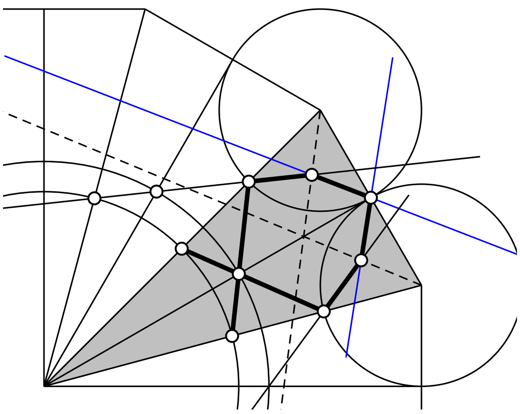


(3)

Construction of a dodecagonal rosette

Suppose that we want to construct a rosette with 12 petals located in the corner of a rectangular contour. First, we divide the right angle into 6 equal parts (1). The lines *a* and *b* crossing the edge of the dodecagon (drawing 1), are the lines connecting rosette petals with the outside pattern. We divide the angles shown in (2) into two equal parts. This can be done with a protractor or a geometric construction. By drawing the two circles (3) we create the outer part of the petal with four equal segments and equal angles (3). By extending one of these segments (4) we create one portion of the rosette (the shaded triangle in (4)). Then we copy the pattern from the triangle into the remaining triangles of the quarter of a regular dodecagon.

This method can be used to create a rosette with any number of petals. Note also how important are the lines connecting the rosette with the outside pattern. They decide how the rosette will look (see 6 and 7).



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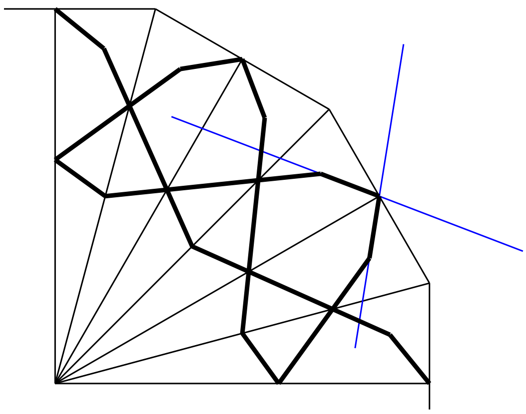

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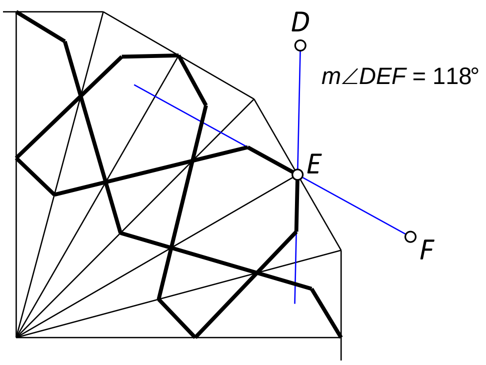

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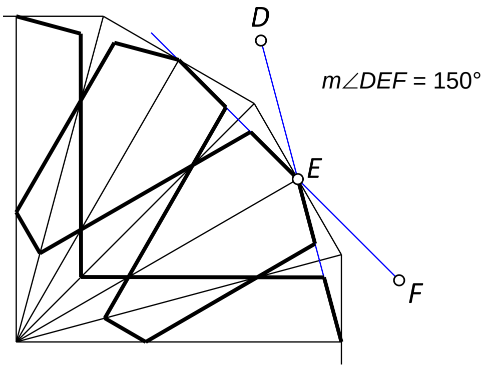
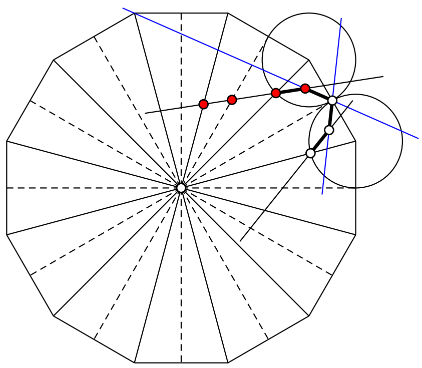


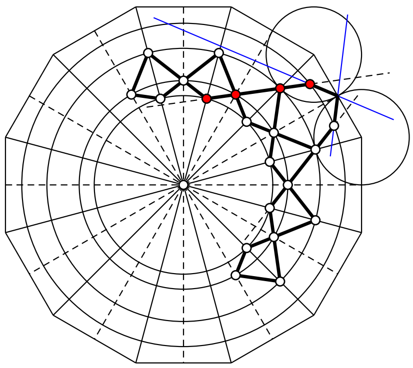
(7)

These two drawings show how the shape of the rosette changes depending on the two lines entering the polygon inscribing the rosette.

While creating a complete rosette it can be more convenient to develop the whole rosette differently.



(3a)

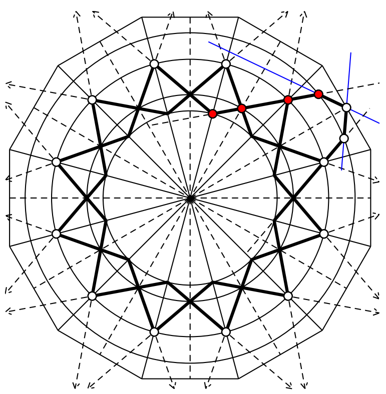


(4a)

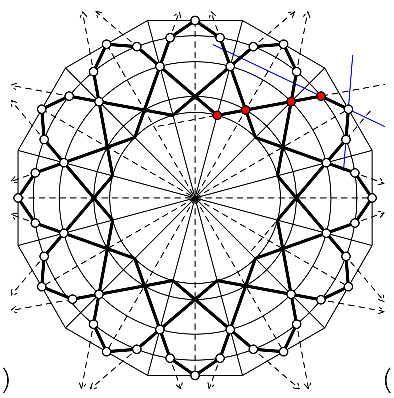
Another way of constructing a rosette

We start from the drawing (3) from the previous page. By splitting dodecagon into long triangles we get 12 triangles. We still leave the points from the drawing (3). Here they are in red (dark in B&W printing). Then we draw four circles through the points created in (3a), and we used them to create a dodecagonal star that is an internal part of the rosette (4a).

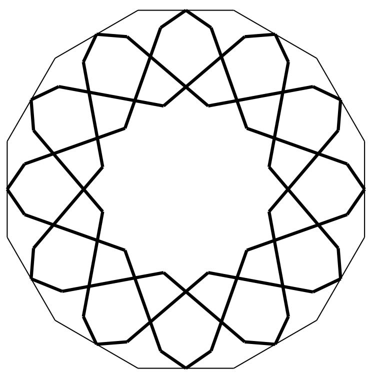
In the drawing (5a) we extend the outside edges of the dodecagonal star. These extensions can be used to draw outer elements of the rosette (6a and 7a).



(5a)



(6a)



(7a)

Designing patterns with regular polygons and shields

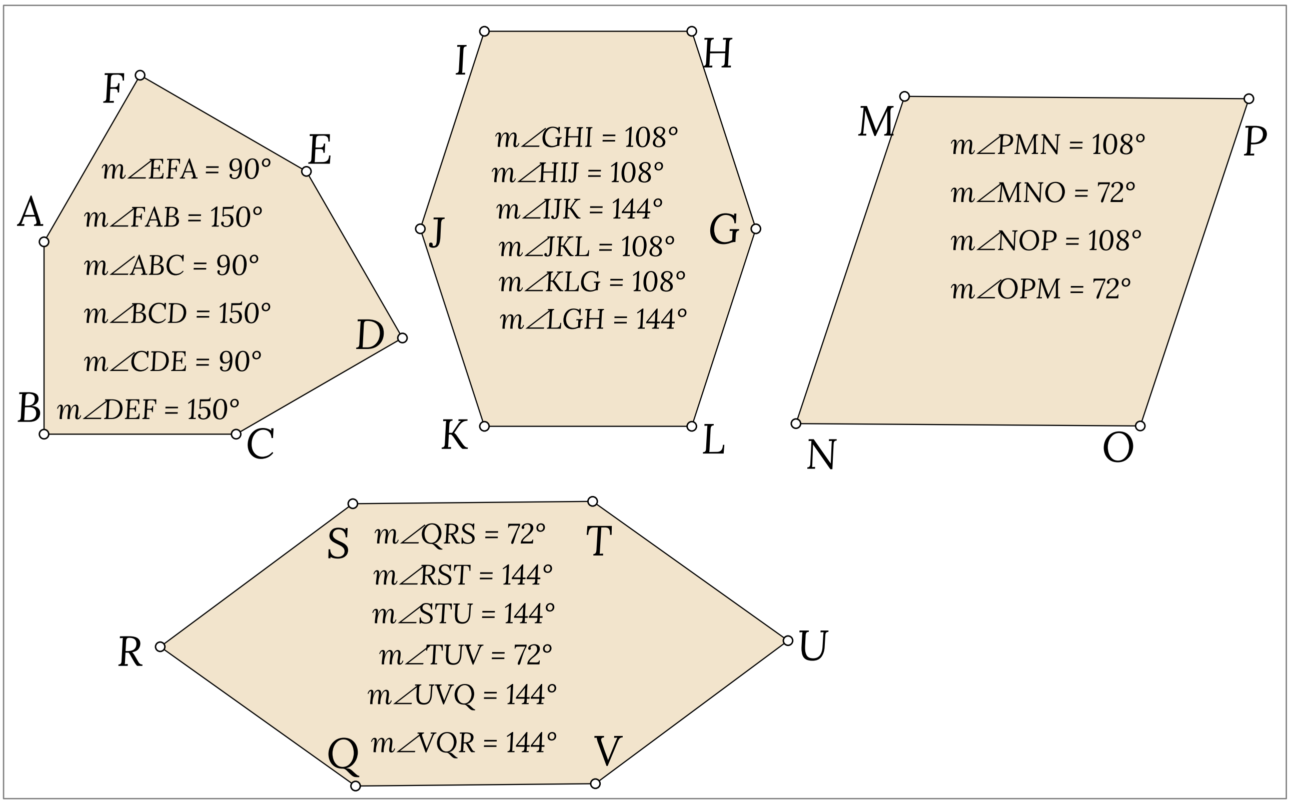
Regular polygons are one of the topics in high school geometry. We know them for centuries. However, we do not know much about shields. Thus let us introduce a semiregular polygon or a shield.

Definition

A semiregular polygon, or a shield, is a polygon with all edges equal and at least two different angles $\alpha_1, \alpha_2, \dots, \alpha_n$ occurring in an organized order:

$$\underbrace{\alpha_1 \dots \alpha_1}_{k_1}, \quad \underbrace{\alpha_2 \dots \alpha_2}_{k_2}, \dots, \quad \underbrace{\alpha_n \dots \alpha_n}_{k_n}; \ \dots; \ \underbrace{\alpha_1 \dots \alpha_1}_{k_1}, \quad \underbrace{\alpha_2 \dots \alpha_2}_{k_2}, \dots, \quad \underbrace{\alpha_n \dots \alpha_n}_{k_n}$$

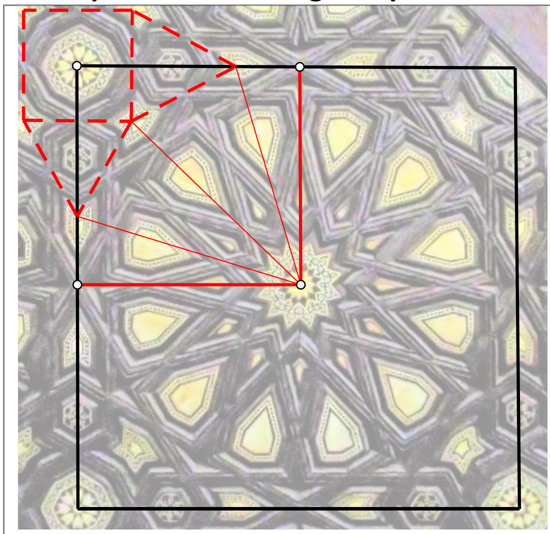
Examples of semiregular polygons (shields):



Here comes a very interesting topic for high school mathematics – investigating the properties of shields. As we know from the history of mathematics, regular polygons were researched by several mathematicians. We can find them in ancient Greek mathematics. They occur also in the works of Johannes Kepler and a few others. In the famous book "Harmonices Mundi" by Johannes Kepler, we can find tessellations with regular and other polygons. In high school textbooks, we can find Pythagorean and Archimedean tessellations with their properties. We can also look at works of contemporary mathematicians. Cumulative information about tessellations with regular polygons can be found on Wikipedia[1]. Although none of these resources mentions tessellations combining regular and semiregular polygons.

[1] https://en.wikipedia.org/wiki/Euclidean_tilings_by_convex_regular_polygons

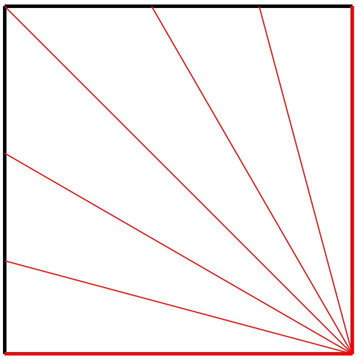
Example 3 – A dodecagonal pattern from Cairo



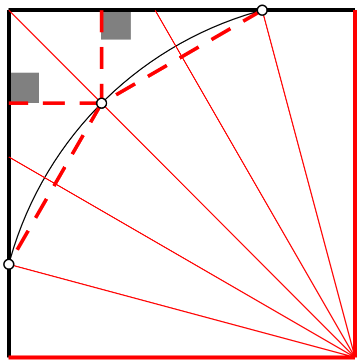
Dodecagonal pattern from Cairo

The presented here pattern comes from one of the mosques in Cairo. This is a very typical example of a design using regular polygons. Here we have a large dodecagonal rosette inscribed into a regular dodecagon, equilateral triangles around the rosette and squares in the corners of the large square (black lines). The square can be used as a contour for the construction of the template. But we can also use only a quarter of this template (between the four shown here points).

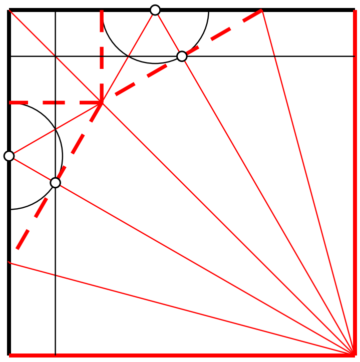
Note also – the rosette is somewhat pushed inward the center of the dodecagon. This is something new in this paper. In the next few drawings, we show the complete construction of this design.



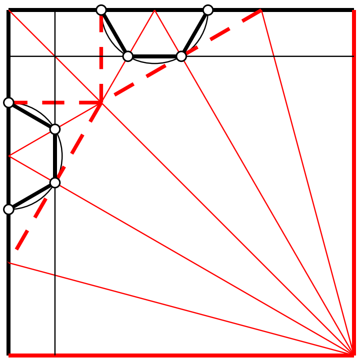
(1)



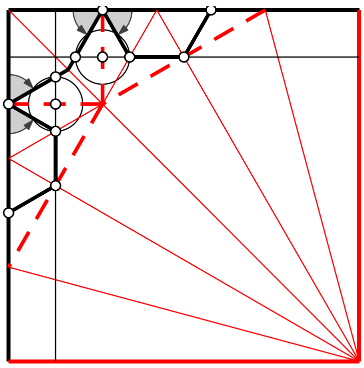
(2)



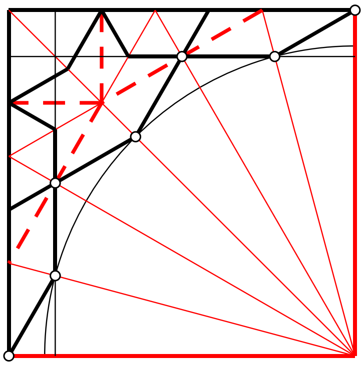
(3)



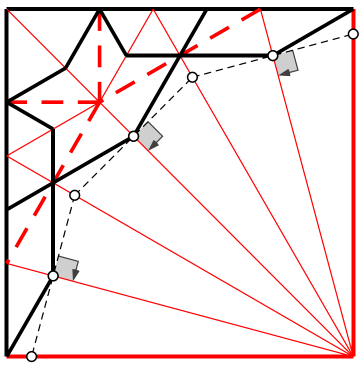
(4)



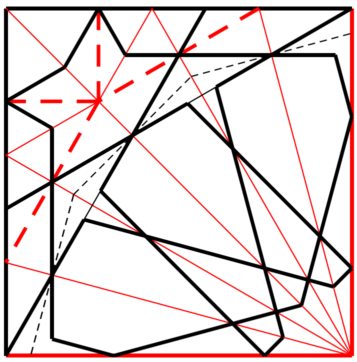
(5)



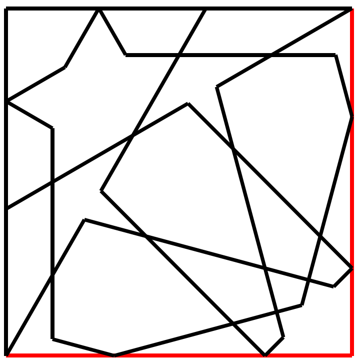
(6)



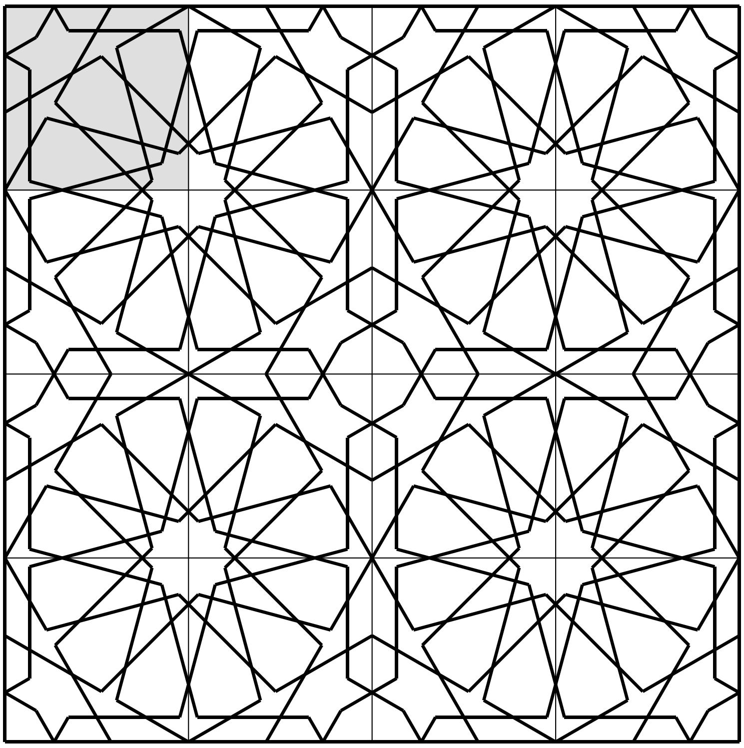
(7)



(8)



(9)



The pattern from a mosque in Cairo

The whole design (left) was created using 16 copies of the template that we developed on the previous page (drawing (9)).

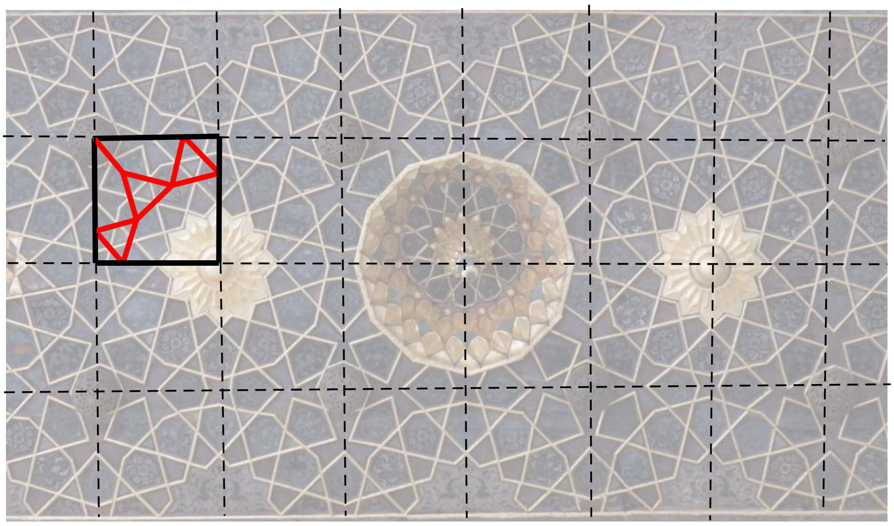
The complete construction process was explained in the drawings (1),..., (9).

The only step that needs a few fords of explanation is shown on the drawings (7) and (8). In (7) we construct a series of triangles inside the quarter of the dodecagon. In (8) we construct a petal of a rosette in the same way as we described it while discussing the construction of rosettes based on regular decagons.

IMPORTANT – note how we cared about having equal angles in multiple places of this construction (see for example drawing (5)). This is essential for not having bent lines.

In the next example, we will show a design using regular polygons and shields.

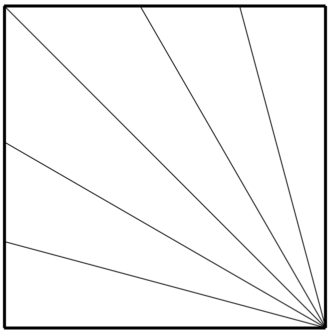
Example 4 – A dodecagonal design from Bursa

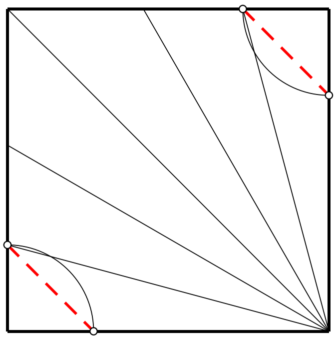


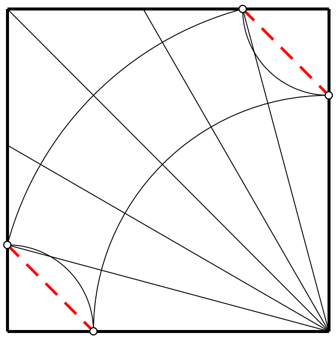
A dodecagonal design from Bursa

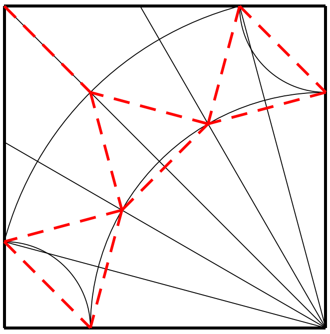
Presented here photograph with lines added in Geometer's Sketchpad shows a large geometric pattern that was created using multiple copies of a square template. The template uses a tessellation with a quarter of a dodecagon, three equilateral triangles, and two halves of a shield. In the top-right and bottom-left of it, we have quarters of two squares.

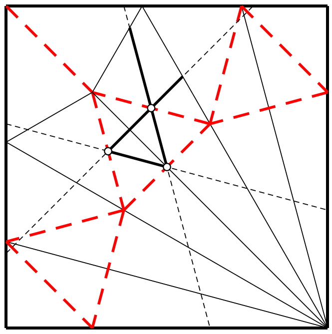
In the next series of drawings, we show how one can construct tessellation with regular polygons and shields and then the pattern. Similar examples we can find in Central Asia and Minor Asia. Several examples of geometric patterns using regular and semiregular polygons are enclosed in Bukhara scroll with architectural drawings. Finally, a good collection of such examples can be seen in some old mosques in old Cairo.

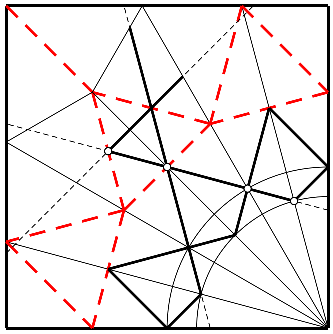

(a)

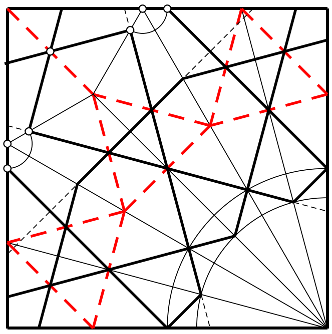

(b)

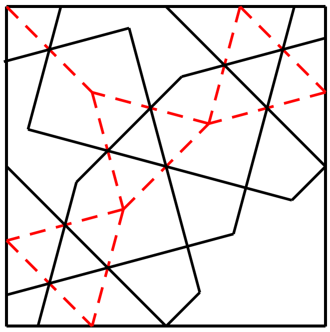

(c)

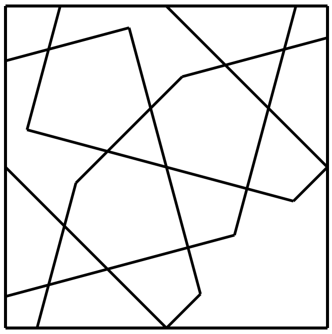

(d)

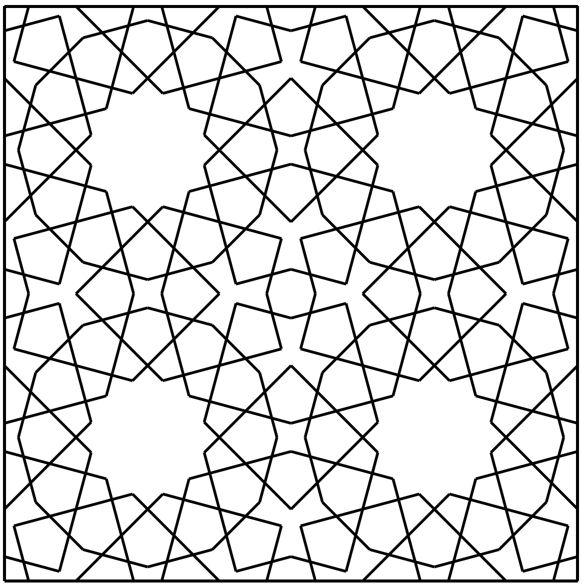

(e)


(f)


(g)


(h)


(i)

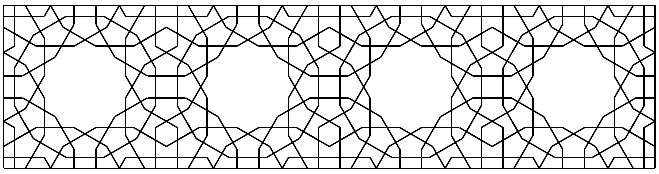


Comments

Drawings in the first row are preparation for creating the tessellation for this pattern. The two dashed segments in (b) are edges of quarters of squares. From them, we get equilateral triangles and other shapes.

Drawing (e) is an essential step for the whole pattern. We just connected the midpoints of edges of one of the triangles. Starting from this moment the rest of the pattern is a simple consequence of this choice and rules of pattern design.

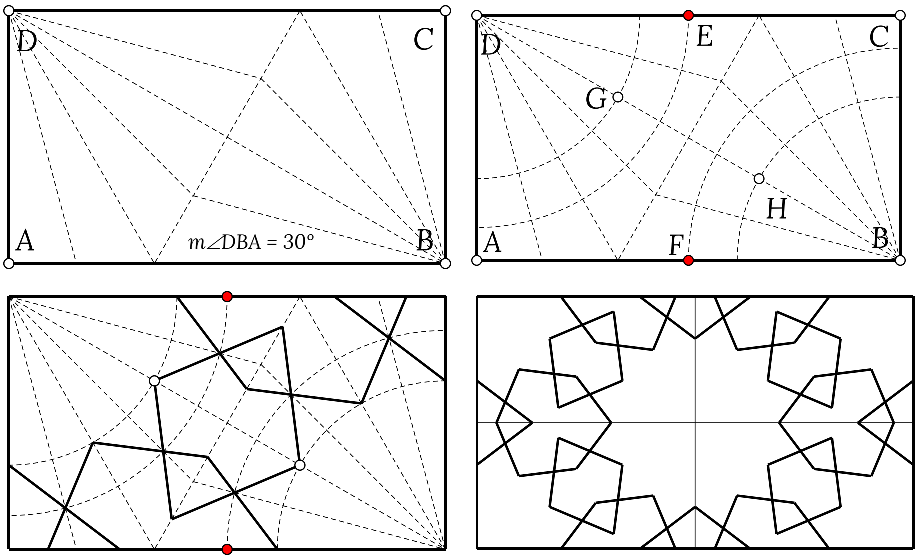
The starting task (drawing (e)) can be different and a new pattern will be obtained (e.g. drawing below).



Direct construction versus tessellation approach

In some papers, we can find constructions of geometric patterns without using a tessellation. In such a case we gain something and we lose something else. Here is an example showing the two ways.

Example 5 – Persian traditional pattern

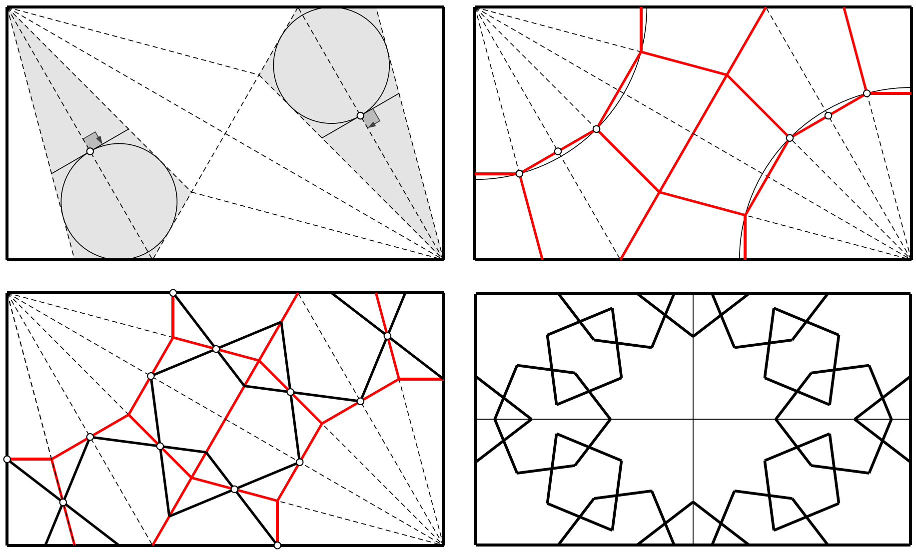


Method 1

In this method we construct the points as follows:

AF=AB/2, EC=DC/2 and DG=GH=HB=DB/3

Points G, H, E, and F are used to draw the four arcs. By joining points on arcs we get the pattern. The whole construction depends on the selection of these four points.



Method 2

In this method, we inscribe circles into shaded kites. We use points of intersection of each circle with the appropriate section line to form two symmetric pentagons. From them, we get a complete tessellation.

The pattern was created by drawing lines through midpoints of edges of pentagons.

By choosing different angles we can obtain a different design.

Conclusions

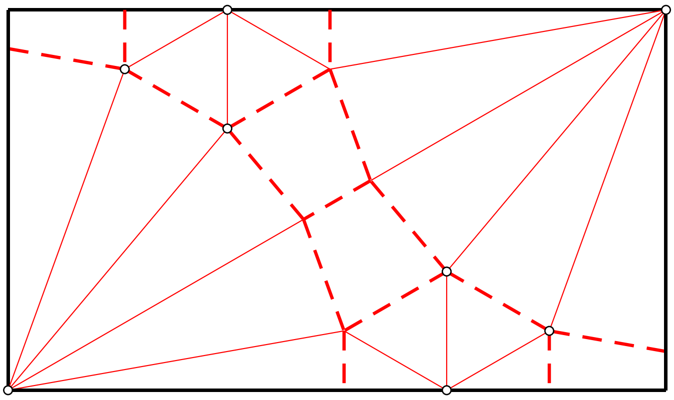
Let us examine the differences between these two methods. In the first one, we have to look for some specific points or tricks to get a pattern. In the second method, we use polygons as close to regular polygons as possible. The pentagons created this way cannot be regular but they are very close to regular polygons. In this case, the geometry of polygons is the foundation of the pattern. In the first case, just a random choice of some secret points gives us the final result.

The final results obtained by using these two methods may have some significant differences.

Breaking rules

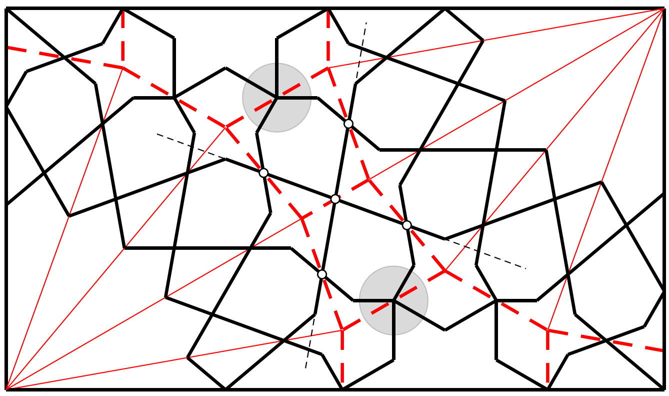
Occasionally we can find a geometric pattern with a construction ignoring one of the rules G1..G7. While discussing gereh rules we did mention a pattern from Bukhara scroll. Below we show the tessellation and the pattern from the Bukhara scroll.

Example 6 – The pattern from Bukhara scroll



The tessellation for the pattern from Bukhara scroll

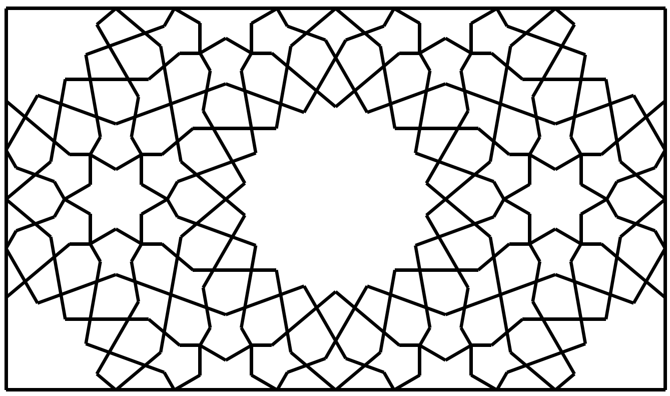
We have here two quarters of 18-gon figures (octadecagons or octakaidecagons) and halves of two hexagons. Points shown here are hints for creating this tessellation. We do this construction in the same way as in example 2. The thin lines are local symmetry lines for hexagons and 18-gons.



The tessellation and pattern from Bukhara scroll

The thin dashed lines are the first lines for the pattern. The whole pattern is a consequence of how these lines were drawn.

The gray circles show places where rule G5 was ignored. Most of us do not like the way how lines inside gray circles are joined.



Larger design using a template from Bukhara scroll

Here we see the final result – a pattern made from four copies of the template from Bukhara scroll.

Summary – connections with mathematics curriculum

While discussing concepts and examples in this paper we frequently passed through issues that bring us to the applied mathematics and mathematics curriculum in every level of mathematics education. It is worth to collect them in one place as we do it now.

1. Geometric constructions – dividing angles and segments into equal parts, constructing perpendicular and parallel lines, constructing regular polygons and shields.
2. Polygons and tessellations – tessellations with regular or symmetric polygons and shields.
3. Symmetries – global and local symmetries of patterns and tessellations.
4. Transformations – translations, rotations, mirror reflections, dilations, and iterations.
5. Connection with technology – all examples discussed here can be developed using traditional drawing techniques with compasses and ruler, or they can be created with geometry software – Geometer's Sketchpad or GeoGebra. Geometer's Sketchpad, in particular, is a very useful tool for pattern designs. It contains a few features that one will appreciate – notebooks with multiple pages, collections of user-made tools, user-made color palettes.
6. Opportunities for students' projects – individual or collective.
7. An axiomatic approach to geometry.

References

[1] Chavey, D. (1989). *Tilings by Regular Polygons—II: A Catalog of Tilings*. Computers & Mathematics with Applications. 17: 147–165.

[2] Day L.F., (1903), *A Book for Students Treating in a Practical Way of the Anatomy, Planning & Evolution of Repeated Ornament*, B. T. Batsford.

[3] Hankin E. H., (1905), *On Some Discoveries of the Methods of Design Employed in Mohammedan Art, Journal of the Society of Arts*, pp. 461-472.

[4] Hankin E. H., (1925), *The Drawing of Geometric Patterns in Saracenic Art*, Memoirs of the Archaeological Survey of India, No. 15, pp. 25 pages and XIV plates, Calcutta, Government of India Central Publication Branch.

[5] Lee T., (1975). *Islamic Star Patterns – Notes*, unpublished manuscript available online as PDF file from http://www.tilingsearch.org/tony/

[6] Majewski. M. (2020). *Practical Geometric Pattern Design: Geometric Patterns from Islamic Art*. Kindle Direct, Independently published (February 10, 2020)

[7] Majewski. M. (2019a). *Practical Geometric Pattern Design – decagonal patterns in Islamic art* (part 1). Istanbul: Istanbul Design Publishing.

[8] Majewski. M. (2019b). *Practical Geometric Pattern Design – decagonal patterns in Islamic art* (part 2). Istanbul: Istanbul Design Publishing.

Appendix – Informal Annotations

There are a few issues that I would like to mention on the margin of this paper. Each of these issues can be a topic for a more in-depth discussion.

Gereh rules – at the time of writing this text we do not know any historical document mentioning these rules. I formulated them in the form given here in 2017 while writing my Polish book on "Gereh – geometry in Islamic art". The motivation for them were observations of ceramic decorations in Samarkand, kundekari examples in Turkey, and some ideas in publications of Russian archeologists from Uzbekistan. These rules are a reflection of the human sense of esthetic. We live in the world where we have an infinity of examples displaying various forms of geometric symmetry – mirror images and flowers with any number of identical petals are the examples of Dn and Cn symmetries, that are literally encoded in our brains. We like objects with predictable behavior of its elements. Thus a line bent or stopped in the wrong place gives us always a feeling that something is wrong. The gereh rules are not given by any supernatural creature, they are not made in iron, and we do not need to follow them exactly in every geometric pattern. We should rather treat them as axioms for a specific group of patterns. In mathematics, we have several similar situations, one such example are the axioms for Euclidean geometry. If you do not like them, then you can always create your own rules or just ignore the tessellation approach. This is exactly what happened in geometry. The axioms for Euclidean geometry were discussed by mathematicians; alternative forms were created, and so-called non-classical geometries found their way to mathematics.

Tessellation approach – the approach presented here one can call a tessellation approach. But, it is important to notice that it does not mean creating geometric patterns like puzzles from predefined tiles. Of course, one can create a set of decorated tiles of specific shapes and start creating patterns using such tiles. The only question is – where are the learning and designing processes. Just look at these two examples from primary school mathematics: (1) *solve equation 2+x=5* and (2) *substitute 3 into the place of x in the equation 2+x=5*. By solving equations we learn how we can solve them. However, by substituting solutions into hundreds of equations we still may not know how to solve these equations. It is important to notice that each geometric pattern was designed for space with specifically given sizes and proportions. Knowledge of this proportion and local symmetries of a pattern will allow us to reconstruct the pattern as well as create hundreds of new geometric patterns. Assembling patterns from decorated tiles is not a designing process. It is only a process of assembling puzzles by using given templates or by a try-and-error method.

References – the list of references for this paper is very limited. I did mention there only publications where geometric, in a mathematical sense, concepts are used. Many other publications discussing geometric patterns cannot be classified into this group. There is at least one book mentioning the mathematical approach to geometric patterns, but the real mathematics is missing in this book. I did not include this book in the references section.

I included, in the references section, the manuscript by Anthony Lee. This is not a mathematical text or a geometric text, it has also some unsure concepts, but a reader can find there many ideas that can be easily transformed into geometric concepts. This manuscript is worth keeping as a source of inspiration.

Summary section – this section contains only geometric concepts that came out from the material provided in this paper. There are a few other geometric concepts worth mentioning that will be discussed in the second part of this paper. These include the existence and use of the golden ratio in geometric patterns, various forms of geometric inflation (yes, there is also inflation in geometry), and several forms of structural geometric design. The majority of known historical geometric patterns are periodic in the mathematical sense. However, in mathematics, we have also the concept of aperiodic patterns. All of this could be discussed later.

Geometer's Sketchpad – I did mention that all illustrations in this paper were created in Geometer's Sketchpad. I am often asked why I prefer to use GSP in my works not another software for geometry or graphics software. There are a few reasons and I have to explain them before again somebody will ask me – why not GeoGebra or Cabri or something else? There are several such reasons – historical and practical nature. The medieval designers used incredibly simple tools and they created marvelous geometric patterns. To understand their difficulties, and consequently, techniques used by them we should use as simple as possible tools. We should use tools that mimic the process of drawing geometric patterns by hand. We should be able to draw lines, circles or arcs, perpendicular and parallel lines, etc. But we should be also able to take a complex drawing and make its reflection through a line or copy it anywhere else the same way as we do with drawings on a paper.

GSP has a few advantages that I have to mention and these advantages do not conflict with the primary requirement – the simplicity of the tool. One of them is the concept of custom tools. In almost every geometry software we have the ability to create custom tools. However, in GSP we can have a library of custom tools grouped into various sections or categories. Thus if I created a complex geometric construction in one of my previous works then I can reuse it in many other designs. I can take also any, even the most complex design, save it as a tool and play it back and forth like a video. This is the feature that I often use while teaching geometric design at the Istanbul Design Center.

Designing with Geometer's Sketchpad is a simple process but it also needs some extra explanations. I will describe it in detail in another paper.

Supplementary materials – I am sure some of the readers of this paper would like to experiment with GSP and concepts present here. You will need an evaluation copy of Geometer's Sketchpad. On the Internet, there are many places with a copy of this software, e.g.

https://www.chartwellyorke.com/sketchpad/x24795.html

Then open the file https://mathandtech.org/eJMT_June_2020/paper2/examples.zip

Unzip the content of the file with examples and experiment with gereh designs provided there.