Geometry, 3D models and Virtual Reality to explain the mathematics of andalusian monumental heritage

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Abstract

In this paper we address the construction of virtual reality scenarios for the NeoTrie VR software, to make some 3D models of architectural objects manageable anywhere. Students will be able to see the pieces (muqarnas, vaults and arches and towers) in a 3D model, manipulate and assemble them, to understand their spatial arrangement. They will also be able to use Neotrie tools to build and thus better assimilate the underlying geometric structures.

1 Introduction

Islamic architecture in Andalusia has provided us with some of the most beautiful architectural objects and intricate geometry. All of these objects have very useful characteristics in the teaching of mathematics, but some are also especially suitable for understanding geometric processes in 3D [7], [8]. This is the case of the muqarnas domes, where hundreds, and sometimes thousands, of pieces are added to form a complex dome given by the union of the pieces. We also deal with crossed ribbed vaults, made for the first time in Córdoba, and which allow the construction of domes with interlocking arches. Finally, some architectures, such as that of the Torre del Oro in Seville, a separate tower that gave welcome along the Guadalquivir river in Seville, are especially suitable for construction in 3D. We will detail these constructions in Section 2.

However, 3D tools are necessary to transfer and understand these concepts with students. Our main goal in this article is to explain the process of designing and creating various virtual reality scenarios with interactive activities.

The chosen tool for this task is NeoTrie VR (briefly, Neotrie) [1], a multiplayer virtual reality software, developed by Virtual Dor (spin-off of the University of Almería), which allow users to create, manipulate and interact with geometric objects and 3D models in general, of different

[1]https://www2.ual.es/neotrie/

types. Before thinking about its implementation in virtual reality, the 3D models (Section 3) were designed for 3D printing to perform manipulative workshops [9].

Recent implementations by the developers team of Neotrie (see [11, 5]) have allowed to generate new scenarios, with friendlier interfaces, menus and new functions, such as arcs creation.

Students can play and learn together the VR scenarios, thanks to the multiplayer mode, which is being improved in the forthcoming Neotrie version 4.7.0. Hence, they will be able to see and interact with the pieces in virtual reality collaboratively, manipulate and assemble them, to understand their spatial arrangement. The VR scenes will also include interactive step-by-step activities to understand the architectural structures by building them themselves using the dynamic geometry tools provided by the software (see Section 4).

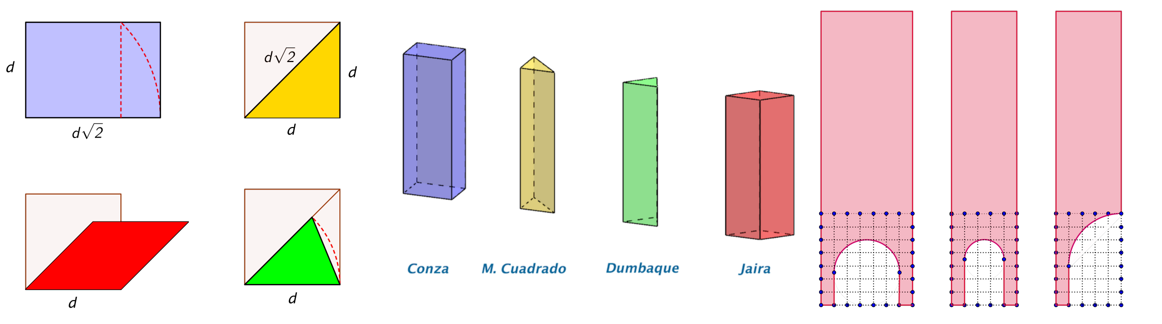
These interactive activities can be better carried out simultaneously with the manipulative workshop with the printed pieces, usually in the same room that hosts the exhibition Paseo *Matemático Al-Ándalus*, as it was done the University of Almería in October 2022, the Instituto Cervantes of Fez in May 2023, and more recently in the Casa Árabe of Cordoba in May 2024.

2 Mathematical description

2.1 Muqarnas mathematical construction

The muqarnas vaults are among the most complex 3D objects in decoration in islamic architecture. They are formed from the aggregation of simpler pieces [4, 6, 12]. Muqarnas are marvelous geometric art creations that challenge our spatial understanding by their capacity of interrelation.

To create them we start from 4 basic prisms, with sections based on $\sqrt{2}$ that fit together (Figure 1a). These prisms are called *conza*, *medio cuadrado* (half square), *dumbaque* and *jaira* (Figure 1b). Other singular pieces such as the *estrella* (star) or the *almendrilla* (little almond) are also added to these. The star usually forms centers with radial symmetry (total or local) around it.



(a) Muqarnas prisms plan. (b) Muqarnas prisms. (c) Convex cutting of a conza.

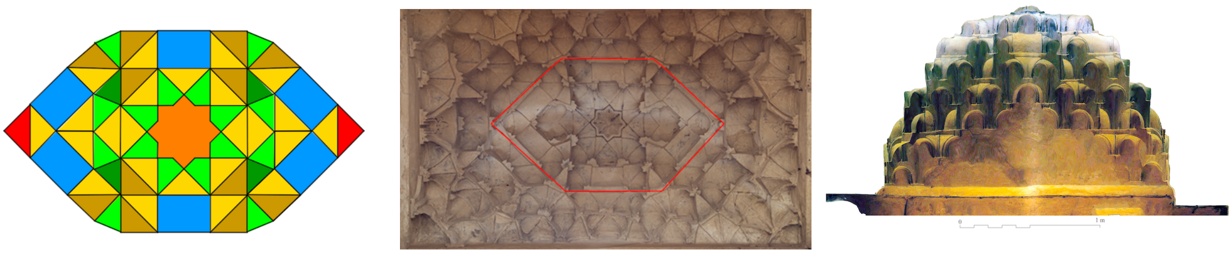
Figure 1: Geometry of muqarnas.

The final pieces or *adarajas* are made by producing some concave cuts of these basic prisms, at one end, in one or both directions. The cuts are based on integer divisions of 7 and 5 (1/7 is very close to $\sqrt{2}$). At the ends of a piece there remain elongated parts of 1/7 of the length or width of the piece, or square, 1/7 x 1/7 size of section. These pins, called *patillas*, are where the small arcs of the cut ends. In Figure 1c we show the *conza* piece with the corresponding cuts.

When the prisms are assembled they form, in plan, a tessellation of a rectangular or polygonal area (see Figure 2a). There are innumerable ways to cover such a space with these prisms, which together with the creativity of the craftsmen, gave rise to a great multitude of possible designs. But all of them maintain properties of radial symmetry around the stars and axial symmetries in the basic axes of the polygon we are dealing with.

After a layer is added around a central piece, new layers are added at lower levels, thus forming level contour lines. A muqarnas dome can have up to dozens of layers at different levels. The patillas or pins are joined together making clusters. This arrangement, together with the concave cuts, makes the whole look very pleasing to the eye. It is a dome structure reminiscent of stalactites in a cave. It is said that the muqarnas domes are inspired by these natural formations (Figure 2c).

In this work we have modeled the central part of the *Puerta del Lagarto* ceiling (XII A.D.), a door in the main mosque of Seville (Spain), over which the present cathedral was built. The original ceiling is the first and simplest built in Andalusia. It has 255 pieces arranged in 7 different levels. (Figure 2b)



(a) Muqarnas vault plan. (b) Picture from bottom. (c) Photogrammetric model showing dome-like structure.

Figure 2: Muqarna vault of *Puerta del Lagarto*.

2.2 The intricate ribbed domes

Another of the pieces that we use in VR scenarios are ribbed domes. These are vaults formed by interlaced arches that cross each other. In the Mosque of Córdoba (VIII A.D.) these domes were tested in a pioneering way. There we can find three different ribbed dome types, although all of them are formed by 8 arches [2].

The first type has 4 crossing points, and in each of them 3 arches of different lengths and curvature cross simultaneously. Although geometrically it is the simplest to construct, it nevertheless has practical problems in its execution, due to the resulting complex crossings. The second type has 8 crossing points, with two overlapping arcs at each point, and where all arcs are equal. It forms in its projection, on a plane parallel to the ground, a stellate polygon *[8/2]* (Schläfli symbol for stellate polygons). It is the most colorful dome and the one that leaves more space in the center, and yet it was not the most replicated in history [3].

This honor goes to the third type of dome, which although geometrically more complicated, it is easier to execute due to the greater number of crossings, which makes the crossing pieces easier to design. It also has 8 equal arcs, but each arc crosses with the 4 consecutive arcs, if these are arranged circularly. It forms, in its projection on a plane, a stellated polygon of type *[8/3]*. In Figure 3 we show the geometrical structure of the three domes. In Figure 4 we show the photogrammetric view of the three types of domes in the Mosque of Córdoba, and in Figure 5 the 3D models for each one of these types.

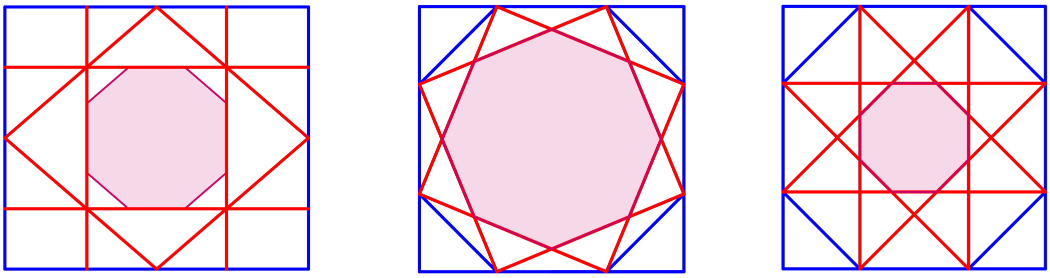


Figure 3: The geometry of the 3 ribbed domes. All with 8 arches, but with 4, 8 and 12 crossing number, respectively: lattice + diamond, and stars [8/2] and [8/3]

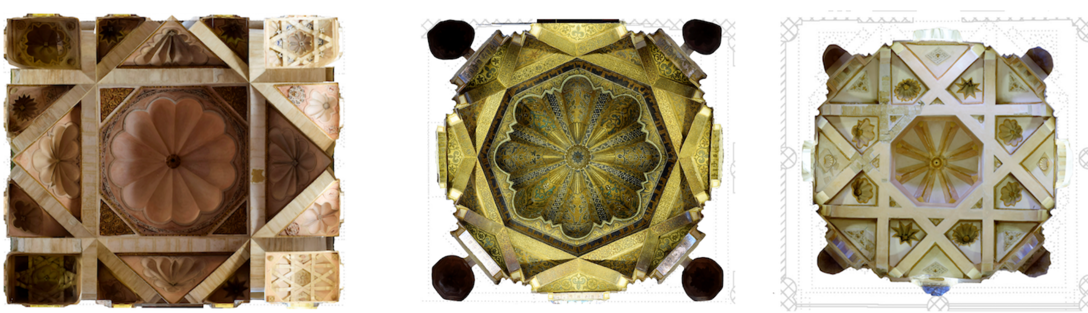


Figure 4: Photogrammetric view of the 3 types of ribbed domes. From left to right: Villaviciosa chapel, Maqsura and lateral Maqsura domes.

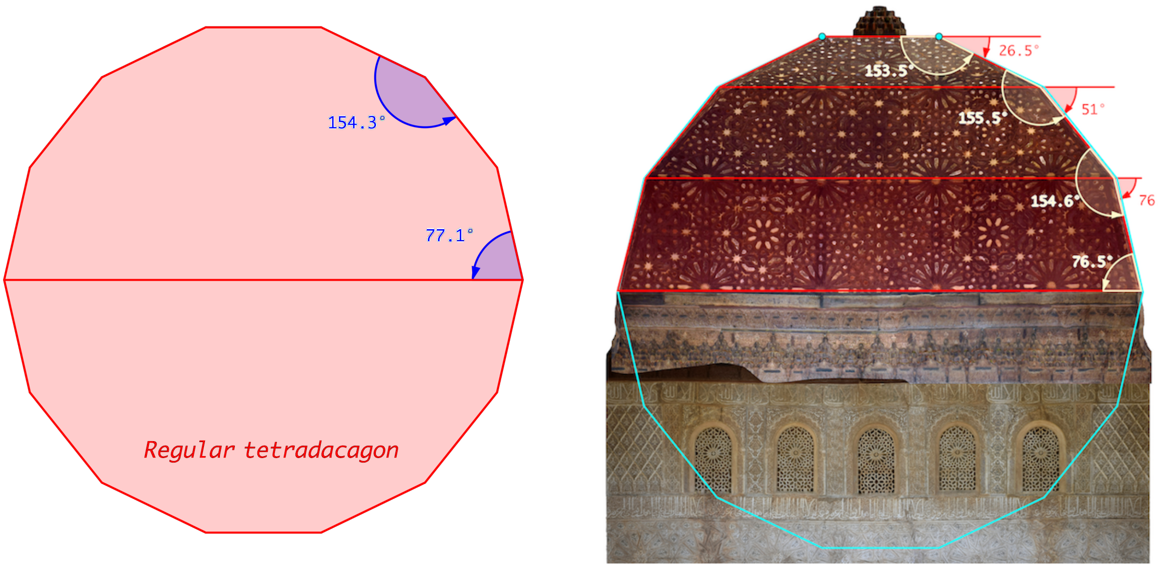
Figure 5: Rendering of the 3 types of ribbed domes in a 3D view.

2.3 The geometry of The Throne Hall

In the third VR scenario we deal with one of the most majestic and beautiful spaces of the Alhambra: the Throne Hall, or Comares Tower. It is the prototype of what has been called *qubba*, in Islamic architecture, formed by a prism with a square base and a splendid wooden roof with 8,017 pieces that resemble a vault of stars. In fact, its symbolic interpretation is that of the 7 heavens of Islamic cosmogony.

This number 7 plays an important role in the geometry of the dome. Therefore, it is not surprising that it is built with a trough-shaped dome, which geometrically corresponds to a truncated and stepped pyramid, whose sections are half of a regular tetradecagon, and then, they have 7 sides. We can geometrically construct the dome using two regular 7-sided semipolygons, resulting from cutting a regular 14-sided one in half. See Figure 6a.

This construction fully corresponds to the design of the dome, as can be seen in Figure 6b. In light blue, a regular 14-sided polygon is represented, built on the side delimited by the two light blue points above. In red you can see how the real construction is barely distinguishable from the geometric one, overlapping exactly on several sides. The angles of inclination of the sides of the dome also coincide, with less than one degree differences between them. This represents not only evidence of the previous geometric design, but also great construction mastery for the 14th century.



(a) Regular 14-gon and their angles

(b) The Throne dome: geometry with superimposed regular 14-gon.

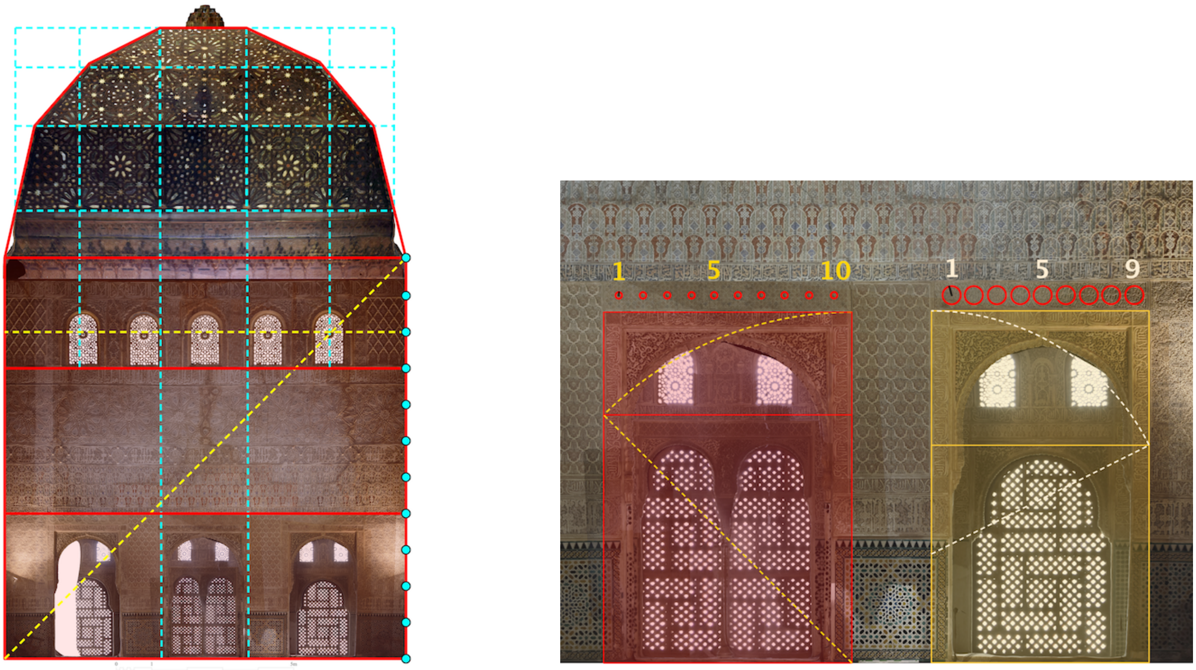
Figure 6: The regular 14-gon and the dome geometry.

To geometrically construct the dome and address an intuitive and accessible didactic procedure in the VR scenario, we proceed as follows:
1st. We cross orthogonally on the upper side, parallel to the ground, these two semipolygons.
2nd. This will result in a first face delimited by the 4 vertices of the sides that intersect in the

center. This will be the upper face, parallel to the ground, which will contain the stars of the seventh heaven, and will delimit the width of the central niche (see Figure 7a).
3rd. Then we pass 4 planes through each of its edges that contain the next side of the base semipolygon. We cut the planes with a line orthogonal to the face of the semipolygon and contained in it, and each plane with their adjacent ones. These will leave us with 4 more faces. This procedure is equivalent to projecting the corresponding sides of the semipolygon, by sliding them throught the straight line that defines each of the 4 sides of the first face.
4th. We continue this procedure for the two remaining sides of the semipolygon on each side, which will give us 2 more faces per side of the room. This way we will fill the 13 resulting faces (3 for each side of the qubba, plus the common central one).

In Figure 7a the main body of the qubba appears, given by a cube with 3 zones delimiting the niches (lower part), plasterwork (central part) and the five windows area (upper part). Above it, a frieze of wooden muqarnas gives way to the dome, which contains 3 trapezoidal faces on each side and the upper closing face. The projection of the grid of unit size (side by side of the semipolygon) is also shown in Figure 7a by dashed light blue lines.

Also, in Figure 7b we obtain $\sqrt{2}$ and golden ratios for the central and lateral niches. We will construct these ratios in the corresponding scenario using the usual geometric method.



(a) The Throne Hall full section. Cube and dome geometry.

(b) Ratios in the niches of the Throne Hall.

Figure 7: The geometry of The Throne Hall.

2.4 The double prisms of Torre del Oro.

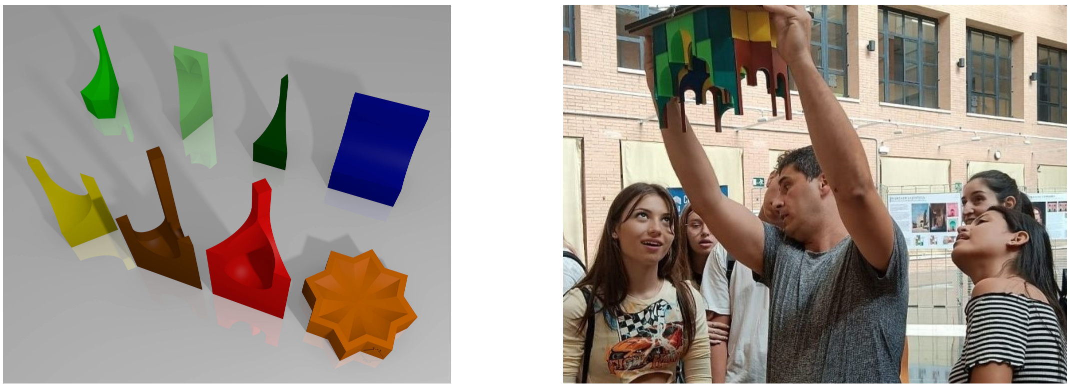
In the fourth stage we address the construction of the *Torre del Oro* in Seville. It is an interesting prismatic tower for the manipulation of prisms and simple 3D constructions. It is the only dodecagonal tower in the medieval islamic world (dodecagonal regular prism).

In it, an external dodecagon (whose diametrical section is $\sqrt{2}$, surrounds a hexagonal prism that contains a spiral staircase inside. The construction of this tower in its spatial form allows us to understand the processes of duplication from a hexagon to a dodecagon and to manipulate them spatially.

3 3D models

Prior to the development of the virtual reality scenarios we modeled all the pieces in order to 3D print them. Those 3D prints have been used to not only ensure that the proportions of the pieces are correct but also to guarantee that they will fit correctly together. Moreover, the pieces have also been used in different workshops with students and the general public where we noticed the interest of the participants in the manipulation of these objects to fully understand their geometrical properties and relations [9]. The modeling have been done using FreeCAD (`https://www.freecad.org`), a free parametric modeler which is quite suitable for this kind of pieces that are well described by its geometrical properties.

For the muqarnas scenario we selected the central part of the muqarnas ceiling of the *Puerta del Lagarto* in the Cathedral of Seville. That section contains 8 different pieces (Figure 8a). All of them part from a simple prism extruded from a rectangle, a star-shaped polygon or a triangle. Then some parts of the prism are cutted out by means of the application of boolean operators with some other basic volumetric parts (cylinders, spheres, cubes...). For example, in Figure 9 one of the muqarnas is constructed from a prism of rectangular base to which a cylinder and another rectangular prism have been substracted. In addition to the muqarnas themselves, a layered box was also designed to help participants to mount the muqarnas ceiling replica that are attached to it by means of some neodymium magnets. In Figure 8b we show the 3D printed muqarnas being used in a practical workshop.



(a) The modeled muqarnas of the central part of the ceiling in the *Puerta del Lagarto*

(b) The 3D printed muqarnas used in a workshop with high-school students.

Figure 8: The muqarnas and their assembly in a workshop.

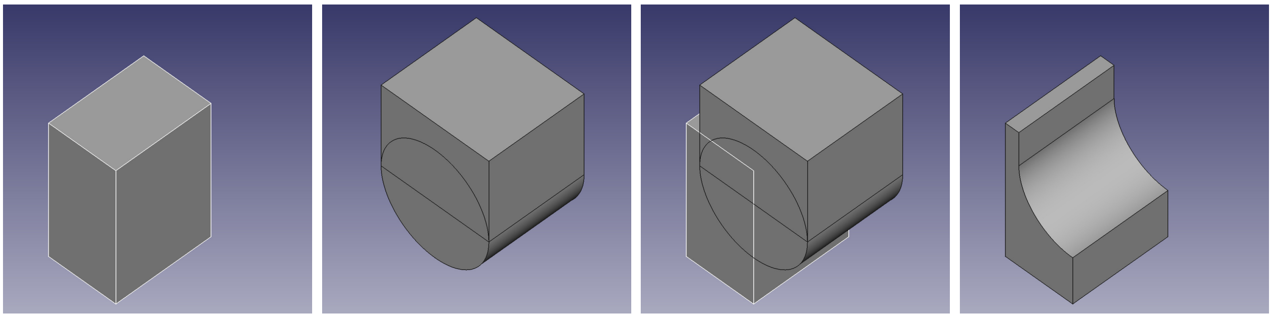
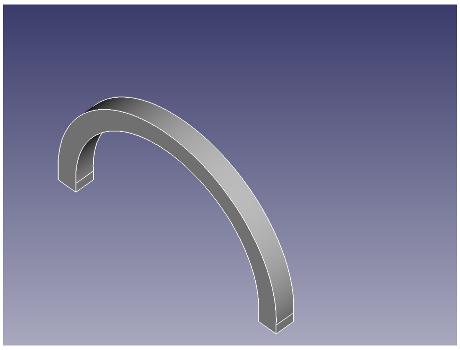
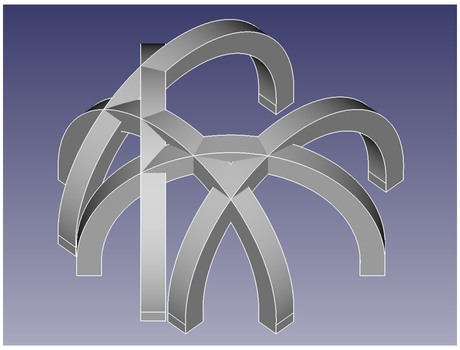


Figure 9: Geometrical modeling of one of the muqarnas.

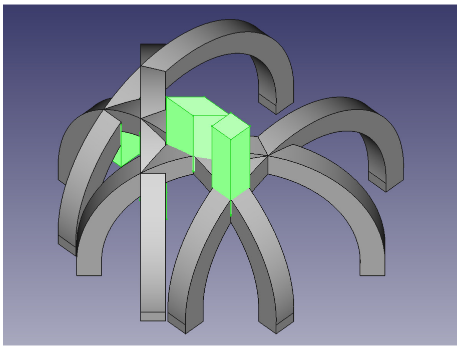
For the crossed ribbed vaults scenario we wanted to model the arches in such a way that 8 pieces of a single model of an arch allow the construction of the ribbed vault. To do so we started with a simple extruded piece from a 2D model of the shape of the arch and then applied different boolean operators to make some cuts and additions in the intersection points of the arches that allow the interlocking of the adjacent arches (Figure 10).



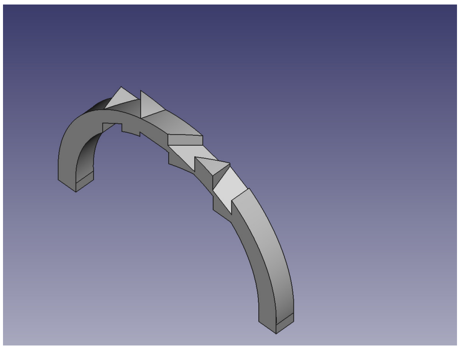
(a) The initial extruded arch from a 2D shape.



(b) The adjacent arches on the vault.



(c) Some prisms are added or subtracted to produce some cuts



(d) Final model of the arch.

Figure 10: Modeling an arch for the crossed ribbed vaults scenario.

In addition to the arches themselves we have also modelled the domes that rest over the arches. This last piece is used in the workshop once the participants have been able to engage all the arches (Figure 11).

Figure 11: Complete model with interlocked arches and upper dome. Picture of the 3D printed model.

4 The VR scenarios

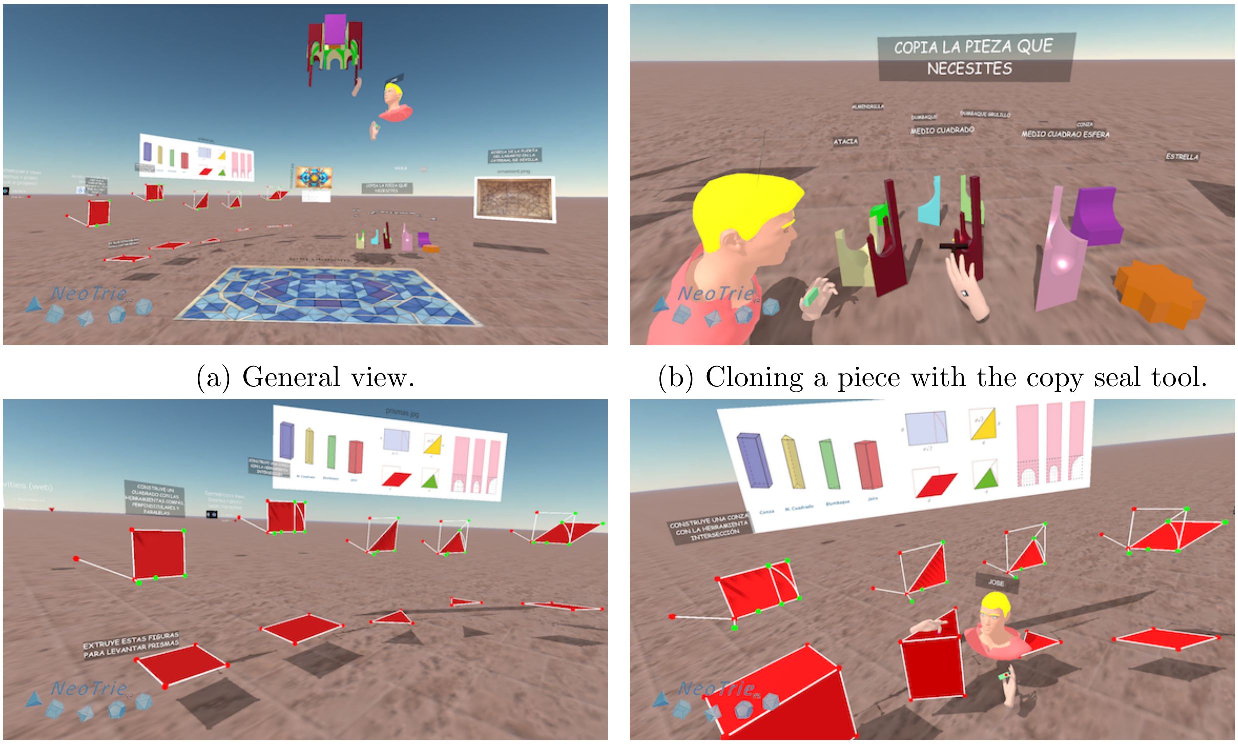
Once in the game, there is a box *Paseo Matemático al-Ándalus* to access the activities of this project. Each scenario contain pictures in this article, short explanations and instructions, the described 3D models (STL format) to assemble, and some interactive activity to work on the geometry associated with the corresponding architectural structures.

The first attempt is usually started with simpler exercises for the user, in which the patterns and organization of the architectural elements are recognized in broad strokes. Later, in a math workshop led by the teacher, students deepen and assimilate the necessary concepts to better understand the geometric structures. To this end, the teacher presents more complex exercises using dynamic geometry tools, such as the compass or perpendiculars and parallels, to build simple models of these pieces and STL structures.

By carrying out these activities the students between the ages of 12 and 16 years work on perpendicularity and parallelism in space, the notions of straight prisms with regular bases, practice and learn about different proportions, build circles and arcs in space, spheres, cylinders and cones, and find a way to scale the figures to obtain models of real sizes.

4.1 Muqarnas

The first scenario (Figure 12) is devoted to muqarnas. Here the player must make copies of the pieces and place them on the virtual ceiling following the pattern indicated on a template (Figures 12b). One can also construct the prisms that generate the muqarnas, by using the compass, perpendicular and parallel tools, and then extruding. However, the software does not yet allow to make spherical or cylindrical cuts on the prisms, which would allow to obtain each of the muqarnas.



(a) General view. (b) Cloning a piece with the copy seal tool.

(c) Build the basis of prisms for muqarnas. (d) Extrude the polygons to get the prisms.

Figure 12: VR scenario for muqarnas dome.

4.2 Vaults and arches

The second scenario (Figure 14) contains the 3 vaults described in Section 2. The player observes the geometrical bases of each vault and makes copies of the arches and put them in place (Figure 14b). Then, one can further propose to students to build the base figure of the dome or to build the missing circled arcs with the compass tool or to build a Carpanel arc.

We would like to point out that the implementation of arcs in space has required the use of shadders in Unity. This has been implemented by Hernandez in the compass tool (see[5, Section 5.2]), which already used the case of a circle passing through 3 points and the circle, giving 4 points, the 4th fixing the normal direction to the circle (Figure 13).

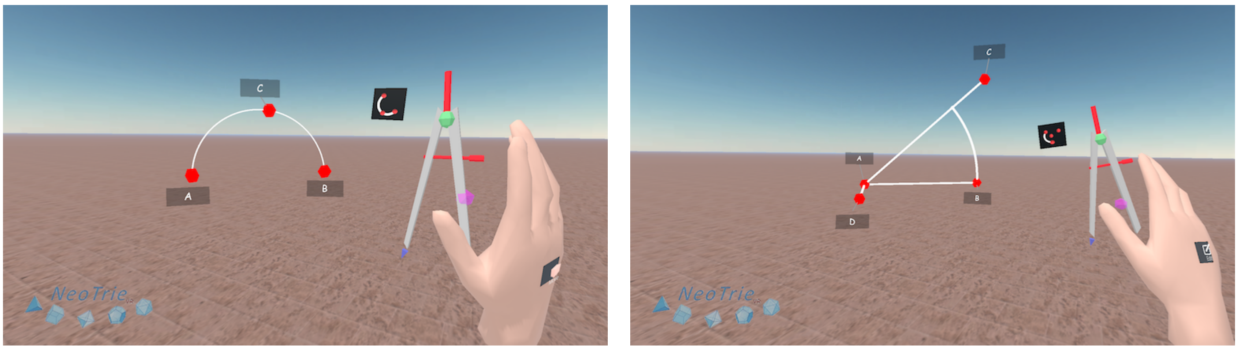
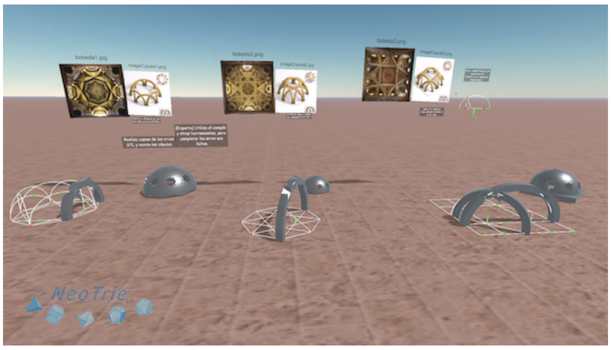


Figure 13: Arc by touching A, B, and C; Oriented arc by touching A, B, C, D.

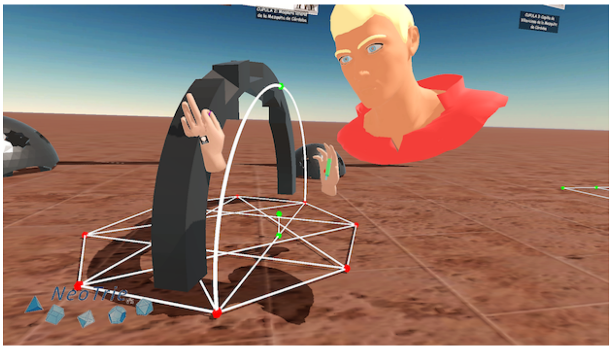
It is also possible to dispense with the fourth point and use the normal obtained by the cross product of two vectors formed by the first 3 points.

The real domes consist of carpanel arches. In the scene the user can see one to try to reproduce it (see Figure 14d). The difficulty in constructing these arcs in space is that we need auxiliary perpendicular directions that indicate the plane where the arc is located, As we have seen in Figure 13.

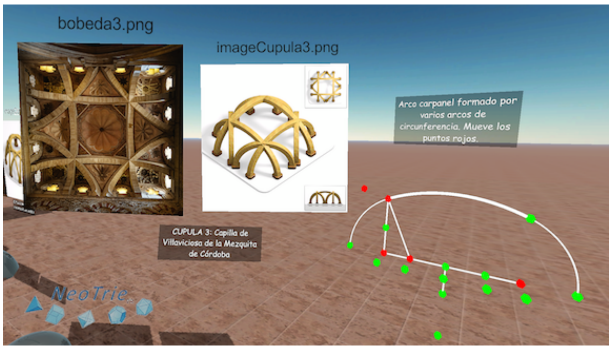


(a) General view.

(b) Assembling a dome.



(c) Make the base figure, the circled arcs and place the STL objects correctly.



(d) Build a dynamic Carpanel arc with the compass, perpendicular and parallel tools.

Figure 14: VR scenario for vaults and arches.

4.3 Torre de Comares

In the third scenario (Figure 16) we used our photogrammetric images to assemble a full-scale model of the throne hall. It is completed with a step-by-step construction of a simple model of its dome, based on the regular tetradecagon adjustment described in section 2.3.

Therefore, students are asked to build a regular tetradecagon in Neotrie. For this, they have to calculate first the angle between two adjacent edges of the regular tetradecagon, that is $180 - 360/14 \simeq 154.2857$, and make the entire tetradecagon with the corresponding tool. Then unlink dependencies of the green points and make a copy to get the regular tetradecagon (see Figure 15).

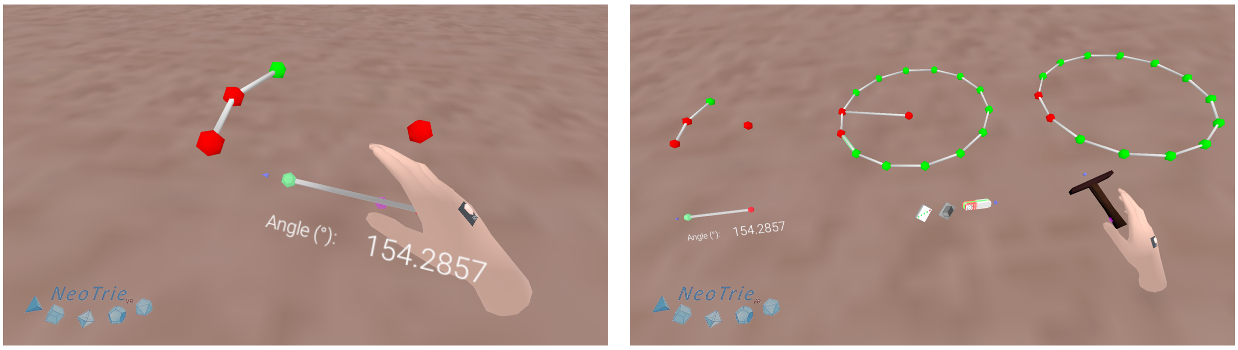
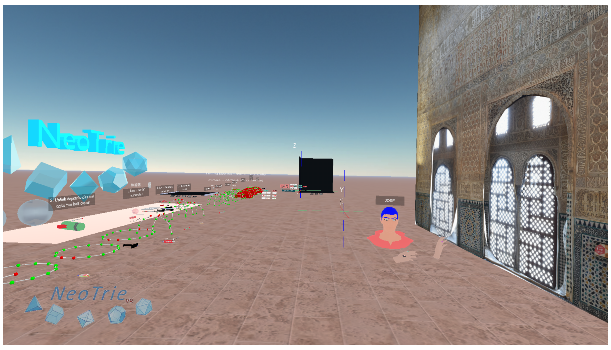
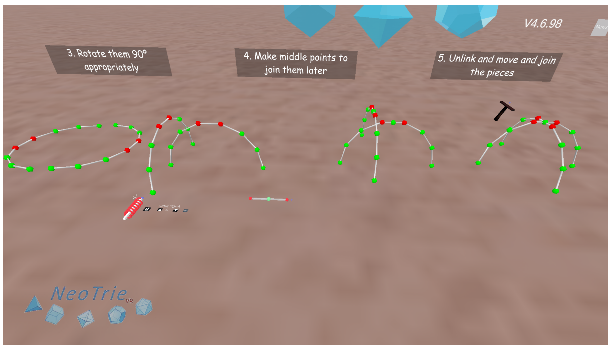


Figure 15: Touch two points of an edge, and a third point to fix the plane containing the angle. Remove dependencies and copy the figure.

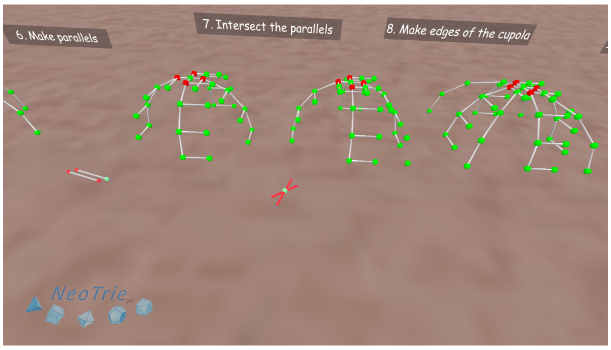
The next step is to use two half parts of the 14-gon to form the axis of the dome (Figure 16b), by rotating them, creating middle points and joining them appropriately. Then, by intersecting parallels, one finds the sided edges of the dome (Figure 16c). Finally, one completes the faces to get a small model of the dome (Figure 16d).



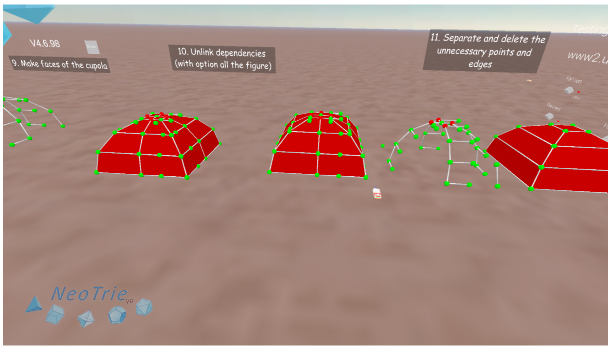
(a) General view.



(b) Building the axis of the dome.



(c) Making the crossing intersecting edges.

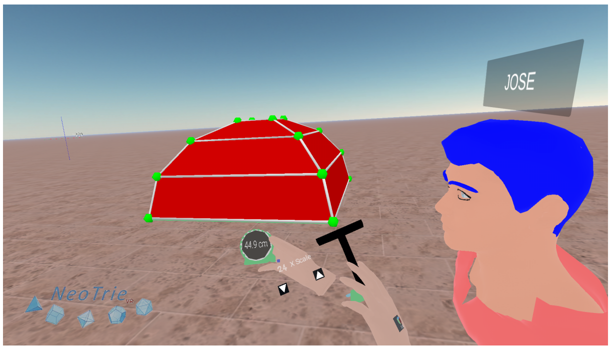


(d) Making the faces and unlink dependencies.

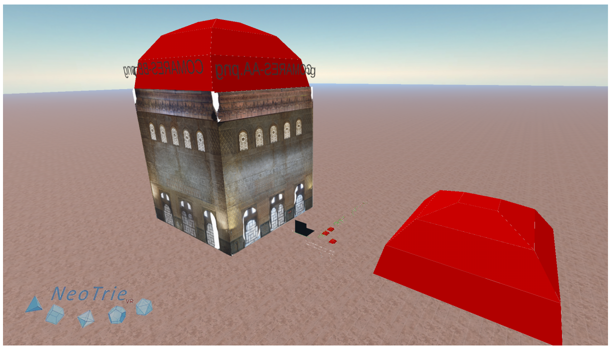
Figure 16: VR construction of a model of the Comares dome.

Students can then get a real scaled copy of the dome. For that, they need first to find the real size of the square base of the tower (11 meters) and measure the length of the base of the small dome (44,9 cm), then get the change of scale as $1100/44{,}9 \simeq 24{,}5$ (Figure 17a), make the scaled copy and put it at the top of the tower (Figure 17b). Furthermore, they can check with the Tape tool that the height of the tower is about 18 meters.

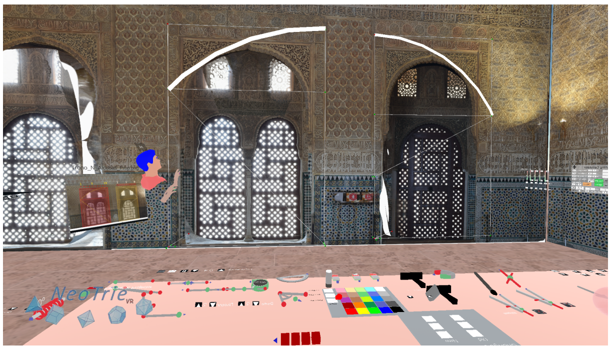
Inside the tower, we in Figure 7b, the points of two interactive rectangles can be moved to find the proportions of two niches of real size (Figure 17c). They can also adjust semicircular arcs passing through 3 points and locate the center (Figure 17d).



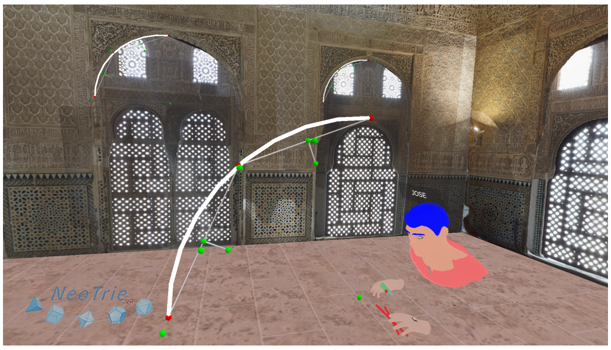
(a) Scaling the dome to get a base of side 11 meters.



(b) Tower mounted with scaled photos and the dome.



(c) Adjusting a $\sqrt{2}$ ratio and golden ratio rectangles on niches.

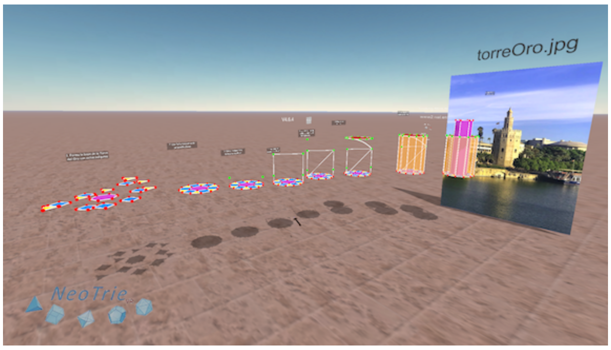


(d) Construction of an arc and its center. Adjustment to the arches of the niches.

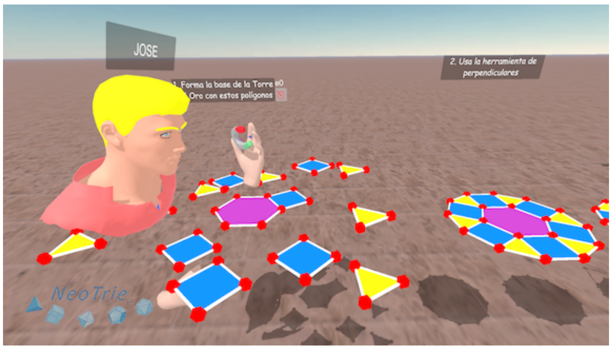
Figure 17: VR model of the tower of Comares with interactive activities.

4.4 Torre del Oro

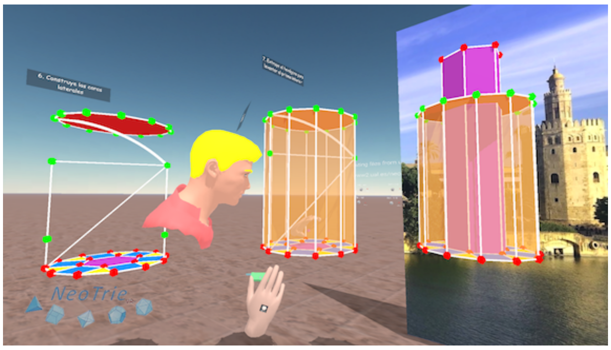
In the fourth scenario (Figure 18) the player can build a simple model of the *Torre del Oro* step by step. Starting with regular polygons (1 hexagon, 6 squares and 6 equilateral triangles), he must use the magnet tool to form the polygonal base of the tower (Figure 18b). Then, lift the prism with the perpendicular, parallel and compass tools, to finish by extruding the inner hexagon. This process can be seen step by step in the video `https://youtu.be/2BkjtrMByNo`. Further exercises could be (see Figures 18c and 18d) to add more geometric elements at the top of the tower: sphere, cylinder, cone, to use the scale copy seal tool to get a model of the tower with real measures and to build a spiral staircase in the interior of the hexagonal prism.



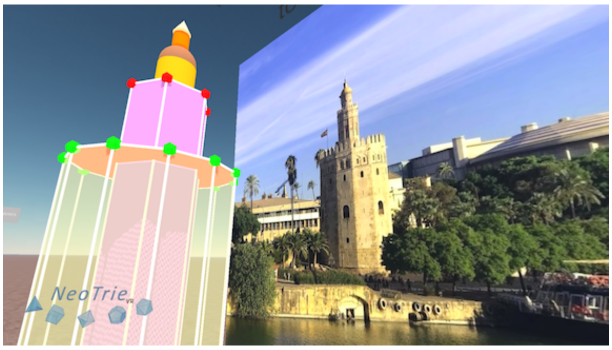
(a) General view.



(b) Assembling the base of the tower.



(c) Last steps of the construction.



(d) Adding more elements to the top.

Figure 18: VR scenario for the Torre del Oro.

5 Conclusion

In this paper we have described a motivating learning situation that combines history, art and geometry, with the help of new technologies (3D printing and virtual reality). Students can learn by building by themselves various structures involving important geometry concepts: scaling, proportions, parallelism, perpendicularity, intersections, symmetries, projections, tessellations, spatial curves and surfaces, etc. They have the incentive of being able to build them in real size, in an interactive and totally immersive way.

We conclude that the power of VR, together with the challenge of its assembly and the admiration aroused by its composition make NeoTrie VR, along with these scenarios, an ideal tool for teaching mathematics together with art and technology, giving rise to a STEAM workshop.

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