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Deconstructing the Constructible Numbers

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Abstract

Constructible numbers are numbers that can obtained using only a straightedge and compass. This paper seeks to unpack the meaning of that statement. We begin with the lattice of integers, providing a discrete geometric foundation for exploring which numbers and coordinates can be produced through successive geometric constructions. We consider some granular questions like constructible distances between integral lattice points and how restricting our set of tools of the number of uses of those tools affects our ability to solve and construct. This all culminates with an extension to higher dimensions and student investigations intended to deepen understanding.

1 Introduction

As mathematics developed in the western world, it slowly shed some of its original geometric motivations and underpinnings. Thus, many students do not connect "squaring" or "cubing" a quantity with its geometric interpretation. In this paper, we will look at how various collections of numbers relate to geometry. In particular, we aim to demonstrate how one can arrive at various real (and complex) numbers via geometric operations. And we will connect some of these ideas to "solvability."

The term *constructible numbers* often refers to enumerated *lengths* constructible with straightedge and compass. In their most simple form, these lengths result from a series of intersections of lines and circles. From a more formal perspective, constructible numbers are often contextualized in the study of the theory of fields (i.e., number systems that allow addition, subtraction, multiplication, and division by nonzero elements), a portion of Galois theory commonly investigated in graduate mathematics. For an overview of constructible numbers for those with some modern algebra background, we recommend [7]. For those more confident in abstract algebra who want a full exposition, we recommend [8].

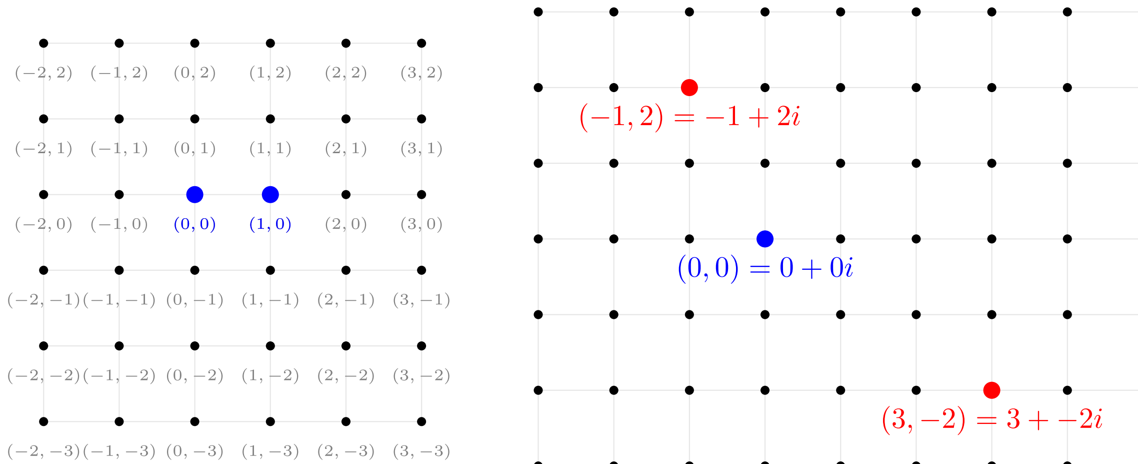

Figure 1: Our initial lattice is $\mathbb{Z}^2 = \{(a, b) \mid a, b \in \mathbb{Z}\} \cong \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$.

2 A Rational Discussion

While the investigation of Constructible Numbers typically begins with the points $(0, 0)$ (the origin) and $(1, 0)$, we begin with a whole lattice of points with integer coordinates (see Figure 1). We later note that one can develop the lattice via compass and straightedge with merely $(0, 0)$ and $(1, 0)$ as initial data.

Before we begin "constructing lengths," we must discuss collections of numbers that make up coordinates of constructible points. To do so, we conveniently identify a point with coordinates (a, b) and the Complex Number $a+bi$. This identification allows one to treat the complex numbers, $\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$, and the real plane, $\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$, as essentially the same mathematical object. In fact, representing complex numbers as points in the plane was one of the mathematical innovations that helped people see the complex numbers as something worthy of study and of practical use, and not just some "imaginary" mathematical construct.

Notice that our starting point for number constructions, $\mathbb{Z}^2$ (points with integer coordinates), is identified with complex numbers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$, called the *Gaussian integers*. Gauss used this set of numbers to understand a host of number theory problems.

Beginning with our initial integral lattice, $\mathbb{Z}^2$, if we add the straightedge as a tool and start drawing lines, we can build new numbers. We will consider any point where two lines intersect as a new constructible point.

Question 1 *Draw a line through two points with integer coordinates. Then draw another such line that intersects the first. What coordinates (i.e., Complex Numbers) can be obtained by intersecting a pair of such lines?*

Let us begin with our integral lattice, $\mathbb{Z}^2$. Using a straightedge to create vertical and horizontal lines, we can construct $\mathbb{Z}^2 = \{(a, b) \mid a, b \in \mathbb{Z}\}$. (*Yes. We expected that big yawn. So, we'd better move on.*)

Notice that the line through (a, b) and $(a + q, b + p)$ has slope $m = p/q$. Thus, given any rational number $m \in \mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\}$ and integer $b \in \mathbb{Z}$, we can draw the line whose equation is $y = mx + b$ (e.g., use points $(0, b)$ and $(0+q, b+p)$). Pick your favorite nonzero rational number m and draw the lines with equations $y = (1/m)x - 1$ and $y = 0$. These intersect when $0 = y = (1/m)x - 1$ so that $x = m$. In other words, we can construct $(m, 0) = m + 0i$. Since we also can construct 0 (just intersect x and y-axes), we can construct *all* rational numbers! (*Yes. All of them.*) By interchanging the roles of x and y, we can see that all points $(0, m) = 0 + mi$ are constructible as well. (*We encourage you to experiment with our Lattice Line Intersections tool hyperlinked below in this footnote.*[1])

So, what does this say so far about solvability? Well, not much, other than for integers a and b (with $a \neq 0$), $x + b = 0$ is solvable in $\mathbb{Z}$ and $ax + b = 0$ is solvable in $\mathbb{Q}$. (*Ok. Wake up. It gets better soon.*)

What if we consider a general point in $\mathbb{Q}^2 = \mathbb{Q}[i]$? This is only slightly trickier. Pick a point in $(p, q) \in \mathbb{Q}^2$ such that $p \neq 0$ (we already know how to construct $(0, q)$). Then we can draw lines whose equations are $y = (q/p)x$ and $y = ((q-1)/p)x + 1$. These lines intersect when $(q/p)x = y = ((q-1)/p)x + 1$. This implies $0 = -x/p + 1$ thus $x = p$. If so, $y = (q/p)p = q$. In other words, $(p, q) = p + qi$ is a point of intersection of a pair of lines constructible from our lattice. In summary, starting with our integer lattice, we can construct *all* points with rational coordinates (i.e., $\mathbb{Q}^2$) simply by intersecting pairs of lines.

Notice the converse is true as well. Lines through our lattice points have equations $y = mx + b$ where m and b are rational. Consider two such equations $y = mx + b$ and $y = nx + c$. Notice that when solving these equations our solution, assuming there is one (i.e., our lines are not parallel), (x, y) will be a point with rational coordinates. In other words, each point in $\mathbb{Q}^2 \cong \mathbb{Q}[i]$ can be constructed by intersecting a single pair of lines through points in $\mathbb{Z}^2 \cong \mathbb{Z}[i]$ and intersecting such lines only yields such points.

Question 2 *Suppose we use $\mathbb{Q}^2 \cong \mathbb{Q}[i]$ as our starting point and start drawing and intersecting lines. Do we get anything new?*

The answer is, "No." Why? If points in $\mathbb{Q}^2$ form lines that intersect, they do so at another point in $\mathbb{Q}^2$. Intersecting lines determined by our integer lattice simply lead to points whose coordinates are rational numbers. We began with a lattice comprised of well-separated integral points, we now have a dense cloud of all points with rational coordinates. One iteration gets us here, so while we have now produced an infinite set including all rational numbers, more iterations give us nothing new. (*But, we want something new. We want irrational numbers. But how?*) Maybe we should measure distances!

Before we measure distances, what do we have regarding solvability? In summary, whether we begin with our integer lattice or with all points with rational coordinates, using a straightedge allows us to solve $ax + b = 0$ (with $a, b \in \mathbb{Q}$ and $a \neq 0$) in $\mathbb{Q}$. (*This is something, but not much progress.*)

[1] A Maple™ tool for investigating this process is available at: https://maple.cloud/app/6216295368753152/Lattice+Line+Intersections?key=E31FA11A660C44BA87EAA0AA2745D8DEBCFA3A9F566941DB9F993F31B719C92A

3 A Lengthy Discussion

Let us return to our integral lattice for a moment. Recall that the distance between (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Seeing a square root appear gives us hope of running into some Irrational Numbers.

Question 3 *What numbers represent distances between lattice points in $\mathbb{Z}^2$? What about points in $\mathbb{Q}^2$?*

Both questions can be reduced to asking what kinds of distances we can produce between a point and the origin. Why? Because the distance between (x_1, y_1) and (x_2, y_2) is the same as the distance between $(0, 0)$ and $(x_1 - x_2, y_1 - y_2)$. Since both collections are closed under taking differences of coordinates, we can simplify our discussion by beginning at the origin.

Working with our integer lattice and beginning at the origin, we get numbers of the form $d = \sqrt{a^2 + b^2}$ where $a, b \in \mathbb{Z}$. In other words, $d^2 = a^2 + b^2$. (*Hmm... Pythagoras?*) This leads us to ask what sums of squares of integers look like. In classic number theory, this is a well understood, but not completely trivial, problem.

Theorem 3.1 *An integer m can be expressed as a sum of squares if and only if given any prime factor congruent to 3 modulo 4 (i.e., its remainder is 3 when divided by 4) appears an even number of times in the factorization of m. (See K. Rosen,* Elementary Number Theory and its applications *Theorem 13.6 on page 498 [9].)*

We begin by considering examples. While $605 = 5 \cdot 11^2$ can be written as a sum of squares (i.e., $605 = 11^2 + 22^2$), $275 = 5^2 \cdot 11$ cannot, since 11 divided by 4 leaves a remainder of 3 and only appears in our factorization an odd number of times. Therefore, $\sqrt{m}$ is the measure of a distance between two points in our integer lattice if and only if m is a non-negative integer and every prime factor of m that is congruent to 3 modulo 4 appears an even number of times in m's factorization. Every integer as well as numbers like $\sqrt{605} = 11\sqrt{5}$ (e.g., the distance between $(0, 0)$ and $(11, 22)$) are such distances. Whereas $\sqrt{275} = 5\sqrt{11}$ is not.

Wait... Are we saying that using our integral lattice some $\sqrt{integer}$ lengths are constructible and some are not? Yes. For instance, while an infinite number of values are constructible (e.g., $\sqrt{1}$, $\sqrt{2}$, $\sqrt{4}$, $\sqrt{5}$, $\sqrt{8}$, $\sqrt{9}$, $\sqrt{10}$), an infinite number are not (e.g., $\sqrt{3}$, $\sqrt{6}$, $\sqrt{7}$,). In fact, equally many $\sqrt{integer}$ are constructible as are not!

Now suppose we are considering distances between points with rational coordinates on our densely crowded cloud, $\mathbb{Q}^2$. These are distances of the form $d = \sqrt{m^2 + n^2}$ where m and n are rational numbers, say $m = a/c$ and $n = b/c$ (we can always give a pair of fractions a common denominator). What rational numbers, say p/q, can be expressed as $m^2 + n^2 = (a^2 + b^2)/c^2$? Since $(a^2 + b^2/c^2) = p/q = pq/q^2$, we want to know if, given integers p and $q \neq 0$, we can find integers a, b, and c such that $pqc^2 = (a^2 + b^2)q^2 = (aq)^2 + (bq)^2$. Recall that every prime factor of pqc^2 that is congruent to 3 modulo 4 must appear an even number of times. Since c is squared (all its prime factors appear an even number of times), this focuses on demanding pq's prime factors comply. In other words, pq must be a sum of squares of integers. Conversely, if $pq = x^2 + y^2$, then $p/q = pq/q^2 = (x^2+y^2)/q^2 = (x/q)^2 + (y/q)^2$. In summary, $\sqrt{p/q}$ represents a distance between two points in $\mathbb{Q}^2$ exactly when $\sqrt{pq}$ represents a distance between two points in $\mathbb{Z}^2$.

Ugh. This means that infinitely many $\sqrt{p/q}$ or $\sqrt{rational}$ are still not constructible. It also means that, for $a, b \in \mathbb{Q}$, $ax^2 + b = 0$ is generally not solvable in $\mathbb{Q}$. (*Still, we have made some progress.*)

4 Circular Reasoning

Let us flip the last section's question on its head.

Question 4 *Given a point in $\mathbb{Z}^2$, what points can we reach when moving an integer distance away? Or, given a point in $\mathbb{Q}^2$, what points can we reach when moving a rational distance away?*

The answer to these questions is we can reach any point on a circle whose center is a point in $\mathbb{Z}^2$ (respectively, $\mathbb{Q}^2$) and whose radius is a non-negative integer (respectively, a rational number). For example, $(\pi, \sqrt{16-\pi^2})$ is distance 4 away from $(0,0)$. (*Ahh. Did you notice our slick trick? We introduced a new tool. In addition to the straightedge, we now have a compass to make circles and arcs. Very sneaky.*)

Now we have not really *constructed* all these points since constructed points have to be determined by intersections. But even so, there are many points we still cannot reach in this way. For example, $(\sqrt{2}, 0)$ is not an integer distance away from any point $(a,b) \in \mathbb{Z}^2$. Why? The distance would be $d = \sqrt{(\sqrt{2}-a)^2 + b^2} = \sqrt{a^2+b^2+2-2\sqrt{2}a}$. If d were an integer, then so would be $d^2 = (a^2+b^2+2) + (-2a)\sqrt{2}$. Without fully justifying this, we necessarily must then have $a = 0$ (so the square root part does not appear).[2] Then $d^2 = b^2 + 2$ and such an equation has no integer solution.[3] It turns out that many (*no, most*) (*no, virtually all*) points, $(\sqrt{integer}, 0)$, are unreachable as an integer distance away from $(0,0)$. Without further exploring this, we can also state that infinitely many points, $(\sqrt{integer}, 0)$, are unreachable when moving a rational distance away from points in $\mathbb{Q}^2$.

Knowing that we cannot reach infinitely many points $(\sqrt{integer}, 0)$ in this manner, we consider more that are reachable by requiring that our point is a integer (respectively, rational) distance away from two lattice points in $\mathbb{Z}^2$ (respectively, $\mathbb{Q}^2$). As seen in Figure 2, this would be tantamount to intersecting a pair of circles whose centers are integral (or rational) lattice points (and which pass through at least one other lattice point). Avoiding the voluminous complexity of fully unpacking this, we can see that the coordinates of points of intersection show up as solutions of a pair of equations of the form: $(x-a)^2 + (y-b)^2 = c^2$ and $(x-p)^2 + (y-q)^2 = r^2$. If we solve such equations (for example, using "allvalues($solve(\{(x-a)^2+(y-b)^2 = c^2, (x-p)^2+(y-q)^2 = r^2\}, \{x,y\})$)" in Maple), then we see

[2] It turns out that 1 and $\sqrt{2}$ are *linearly independent* when working over the rational numbers, so $x + y\sqrt{2} = v + w\sqrt{2}$ (for rational numbers x, y, v, w) implies $x = v$ and $y = w$. In our equation, we had $d^2 + 0\sqrt{2}$ =interger+$(-2a)\sqrt{2}$ so $0 = -2a$.

[3] Why? Divide any integer by 4 and keep the remainder (i.e., work "mod 4"). One gets 0, 1, 2, or 3. Squaring, one gets $0^2 = 0$, $1^2 = 1$, $2^2 = 4 = 0$ (mod 4), or $3^2 = 9 = 1$ (mod 4). Squares are either equal or differ by 1 (working mod 4). Thus squares of integers can never differ by 2.

that solutions are of the form $rational + \sqrt{rational}$. As for which rational numbers can appear in such a solution, this depends on whether we allow integer or rational coordinates as starting points.

Question 5 *Given two points in $\mathbb{Z}^2$, draw a circle centered at one of those points which passes through the other point. Next, draw a line passing through two points in $\mathbb{Z}^2$. Assuming the circle and line intersect, what kinds of coordinates does such a point of intersection have?*

While this is again too complex to fully unpack, once again we see that points of intersection have coordinates which solve equations of the form: $(x-a)^2+(y-b)^2=c^2$ and $px+qy=r$. It turns out that we still get numbers of the form $rational + \sqrt{rational}$ (for some but not all rational numbers).

Regarding solvability, we can solve $ax+b=0$ for $a,b \in \mathbb{Q}$ (with $a \neq 0$) and we can solve *some* equations of the form $ax^2+b=0$. But given $a,b \in \mathbb{Q}$ (with $a \neq 0$), $ax^2+b=0$ may still require solutions that are out of our reach.

5 Repeat

The next natural thing to consider is repeating our operations. We can draw lines and circles through previously constructed points and consider intersections of such to be new constructed points. It turns out that several of our previous questions were too granular, too detailed. If we allow ourselves to freely iterate, we can more readily see what kinds of numbers we can construct.

When defining constructible points and numbers, one usually starts with just two points (not a whole lattice) and calls them $(0,0)$ (the origin) and $(1,0)$. We then proceed to use two tools, a compass and a straightedge, to build up new points as intersections of circles and lines. If we draw a line with a straightedge through these initial points and then draw a circle centered at $(0,0)$ and passing through $(1,0)$, using a compass, we will get two points of intersection – one of them new! These are $(\pm 1,0)$. By then drawing circles centered at $(\pm 1,0)$ and passing through $(0,0)$, we can construct $(\pm 2,0)$ (see Figure 2). It should not be difficult to imagine that we can now construct all points on the horizontal axis whose coordinate is an integer.

Another standard construction allows us to raise a perpendicular to a line that passes through a point on that line (see Figure 2). This allows us to create a vertical axis which intersects with our first circle at $(0,\pm 1)$. We can then repeatedly draw circles and get all points on the vertical axis whose coordinates are integers. Creating lines perpendicular to axes through these points allows us to construct all of our lattice points in $\mathbb{Z}^2$. So with the aid of our two tools (i.e., straightedge and compass) and multiple iterations, we need only two points to begin our process.

In summary, beginning with only the two points $(0,0)$ and $(1,0)$ and using a straightedge and compass (allowing us to construct parallel and perpendicular lines), we can build up the entire lattice $\mathbb{Z}^2$... and so much more. In fact, we can now construct $\sqrt{z}$, where z is any non-negative integer. See Figure 3 (*and experiment with our our hyperlinked Iterated Integer Square Roots tool in the footnote*[4]). Notice using

[4] Use Maple to animate Figure 3: https://maple.cloud/app/6198229031321600/Iterated+Integer+Square+Roots?key=6C99D4D7477B447B851C8C88FC22C5B33ED6119483064DA1A7110AAC5C2C91E6

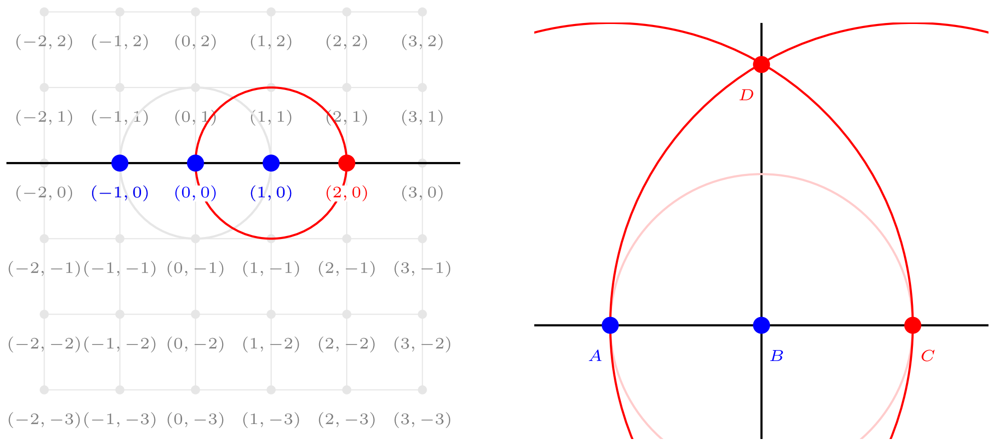

Figure 2: Constructing $2 \in \mathbb{Z}$ from 0 and 1 and Raising a Perpendicular.

the (constructible) lines $y = 0$ and $y = 1$ and drawing a circle (centered at $(0, 0)$) passing through $(1, 1)$, we construct $\sqrt{1^2 + 1^2} = \sqrt{2}$ on the horizontal axis $y = 0$. Then, raising a perpendicular we can construct $(\sqrt{2}, 1)$. Drawing a circle (centered at $(0, 0)$) passing through that point, we construct $\sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3}$ on the horizontal axis. Continuing this process, we get $\sqrt{3}$, $\sqrt{4}$, etc. (*Again, we are making more progress, filling in previous gaps like* $\sqrt{3}$, $\sqrt{6}$, $\sqrt{7}$, *etc.)*

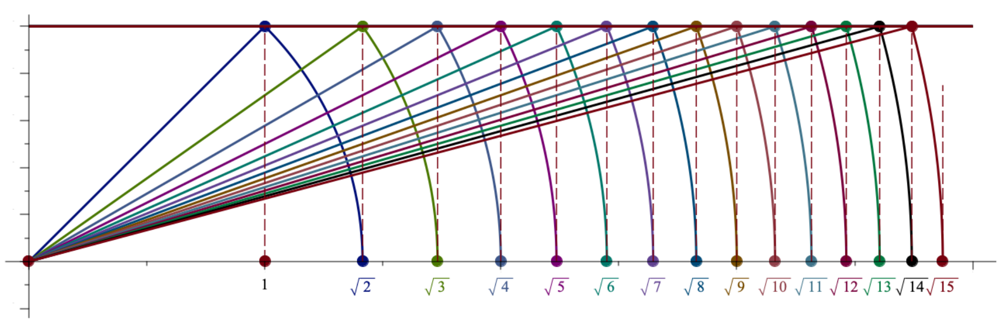

Figure 3: Constructing Square Roots by Iterating.

Let us re-identify our points (a, b) in $\mathbb{Z}^2$ with complex numbers $z = a + bi$ in $\mathbb{Z}[i]$ (the Gaussian Integers). Given two complex numbers $z = a + bi$ and $w = c + di$, standard constructions allow us to construct $z + w$, $z - w$, zw, and z/w (if $w \neq 0$). This means that our collection of constructible numbers is closed under addition, subtraction, multiplication, and division (not by zero). The technical name for such a number system is a *field*. For example, the rational, real, and complex numbers are fields but the

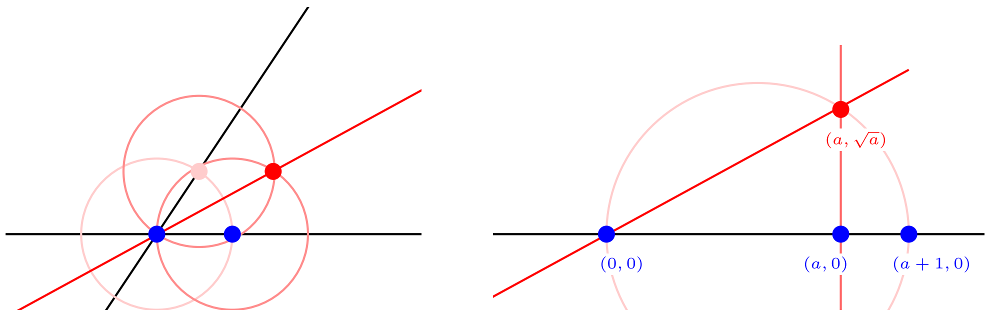

Figure 4: Bisecting an Angle and Computing a Square Root.

integers (lacking division) are not.

Now if we are limited to the operations of addition, subtraction, multiplication, and division, and we started with $(1, 0) = 1 + 0i = 1$, it would result in producing only the rational numbers $\mathbb{Q}$. If we also included $i = \sqrt{-1}$, we could produce $\mathbb{Q}[i] = \{a + bi \mid a, b \in \mathbb{Q}\}$.[5] When we only allowed ourselves to draw and intersect lines (starting with our lattice $\mathbb{Z}^2 \cong \mathbb{Z}[i]$), this was our resulting collection of numbers: $\mathbb{Q}^2 \cong \mathbb{Q}[i]$.

However, constructible numbers allow another operation that spills outside rational numbers: we can take square roots. For those unfamiliar with square roots of complex numbers, we first put our complex number in *polar form* (just like polar coordinates): $z = r\cos(\theta) + r\sin(\theta)i$ so that r (i.e., the modulus) is the distance from the origin to z and θ (i.e., the argument) is the angle swept out by z. Then the square roots of z are $\pm\left(\sqrt{r}\cos\left(\frac{\theta}{2}\right) + \sqrt{r}\sin\left(\frac{\theta}{2}\right)i\right)$. To determine a square root, we need to be able to bisect an angle, θ, and find the square root of a non-negative real number, r. There are standard constructions to accomplish these tasks (see Figure 4). Now, our collection of constructible numbers is also closed under square roots (even for negative quantities and stranger complex numbers).

We now have access to a world of new numbers such as $1 - 4\sqrt{3}$ and $\sqrt{\dfrac{2 + \sqrt{123}}{5}} + \sqrt{-17}$. Using rationals as a starting point (or we could just use 0 and 1), any number built from a finite number of steps of addition, subtraction, multiplication, division (not by zero), and taking square roots is constructible. The collection of all such numbers is the *field of constructible numbers*. If we intersect this collection with the real numbers (i.e., project our points onto the horizontal axis), we get the *field of real constructible numbers*.

Question 6 *Does this give us all numbers? Can we get numbers like cube roots or* π*?*

In the nineteenth century, as abstract algebra developed and studied fields of numbers, it clarified exactly what numbers could or could not be constructed. Since our constructions build from the rational operations and square roots, any constructible number is the root of a polynomial with integer coefficients. In fact, it must be a root of such a polynomial whose degree is a power of 2.

[5] This field, $\mathbb{Q}[i]$, is the field of fractions of the Gaussian integers, $\mathbb{Z}[i]$.

Providing a little more detail, let us consider how we are constructing our points (identified with complex numbers). Given already constructed points, we draw lines through those points and circles centered at one such point and passing through another. Equations of such lines and circles are of the forms $ax + by = c$ and $(x - a)^2 + (y - b)^2 = c^2$ where a, b, and c are previously constructed numbers. Without writing down all of the details, notice that intersections are solutions of linear or quadratic equations such as $az^2 + bz + c = 0$ with a, b, and c constructible. It turns out that roots of such quadratic equations coincide with roots of polynomials whose coefficients are integers and whose degree is a power of 2 (e.g., degree 1, 2, 4, 8, etc.). This is most *easily* (but not necessarily *simply*) explained using some mathematical field theory. Again, we point the interested reader with some abstract algebra background to [7] or [8].

Immediately, transcendental numbers like π and e are definitely off the table as constructible numbers. Why? Transcendental numbers are, by definition, numbers that are not roots of *any* nonzero polynomial with integer coefficients.[6] Putting this together, we have ruled out *squaring the circle*. Specifically, given a square of area 1, we cannot construct a circle of area 1. Why? We would need to be able to construct a circle with radius $\sqrt{\pi}$. Since π and thus $\sqrt{\pi}$ are transcendental, this construction is impossible.

Algebraically, intersecting lines and circles amounts to solving linear and quadratic equations. Therefore, constructible numbers cannot access the world of cube roots and other more complicated things. (*Umm... For now. We will later return to this for more consideration.*) Another ancient problem asks one to *double the cube*: given a cube of volume 1, construct a cube of twice the volume (i.e., volume 2). However, doing so would require us to construct such a cube's side of length $\sqrt[3]{2}$. Such a number is not constructible since, while it is the root of a integer coefficient polynomial of degree 3 (i.e., $x^3 - 2$), it is not a root of an integer coefficient polynomial whose degree is a power of 2.

It turns out that while we can extract square roots (and repeating can extract fourth, eighth, etc. roots), these constructions can never extract a root of an irreducible cubic polynomial. In particular, they cannot construct cube roots of integers, unless they are perfect cubes (e.g., $\sqrt[3]{8} = 2$). Likewise, we cannot extract third, fifth, sixth, seventh, etc. roots.

Related to this, while we can bisect any given angle, we cannot trisect all angles. In particular, the $60°$ angle is constructible (consider a special triangle to see why), but it is impossible to construct a $20°$ angle. It turns out that if we could, we could construct the numbers $\cos(20°)$ and $\sin(20°)$, but these are roots of irreducible cubics with integer coefficients (and thus not constructible).

In the end, constructible numbers can look quite complicated, like $\sqrt[8]{\frac{\sqrt{3}}{5} - \sqrt[16]{321}}$, but they are still pretty limited when we consider the vast scope of all real or complex numbers. In fact, the constructible numbers are a subfield of all algebraic numbers (i.e., numbers that are roots of nonzero polynomials with integer coefficients). This is because constructible numbers are roots of polynomials (with integer coefficients) whose degree is a power of 2 and algebraic numbers are roots of a polynomials of *ANY* degree. Nevertheless, both the algebraic and constructible numbers are infinite collections but only countably infinite. On the other hand, the real and complex numbers are uncountable (specifically continuum in

[6]We point the interested reader to [5] for more about transcendental numbers. Alternatively, consider [4] for an elementary introduction to transcendental numbers and [2] for an explanation of why functions such as sine and the natural logarithm are called transcendental functions.

cardinality).[7] This means most real and complex numbers are inaccessible in terms of our constructions.

Returning to solvability, constructible numbers allow us to solve linear and quadratic equation with constructible coefficients. Ugly equations like $\left(\sqrt{1+\frac{3+\sqrt{5}}{2}}\right)x^2+\left(16-\sqrt[4]{1+\frac{\sqrt{3}}{4}}\right)x+\sqrt[8]{53}=0$ are solvable in the field of constructible numbers. However, a simple equation like $x^3-3=0$ is not!!!

Ahhh, but why? Why are things as simple as cube roots not constructible in the constructible numbers? On the most simple and intuitive level, let's just say that, using a compass and straightedge, these numbers are constructed "on the flat" or in a plane. Think about the limitations of a flat square on the page.[8] However, cubed numbers and cubed roots would most intuitively be contextualized in 3-dimensional space and working with polynomials of degree 3.

6 The Long and Short of It (i.e., A Summary)

Through both review and extension, we provide some notes regarding some of our current findings (points 1–3). We then mention that how our findings generalize to higher dimensions (points 4–6).

1. Intersections of lines through points in the lattice $\mathbb{Z}^2$ construct *all* points in a dense cloud of points with rational coordinates $\mathbb{Q}^2$.

2. In our lattice of points with integer coordinates, $\mathbb{Z}^2$, restricting line segments to vertices at points $(integer, integer)$, *some* $\sqrt{integer}$ lengths are constructible.

3. Beginning with our lattice of points with integer coordinates and allowing iterative constructions, *all* $\sqrt{rational}$ lengths are constructible.

4. If we move to higher dimensions, intersecting (hyper-)planes through points with integer coordinates (i.e., hyper-planes through points in the lattice $\mathbb{Z}^n$) will yield points with rational coordinates (i.e., points in $\mathbb{Q}^n$).

5. Lengths of line segments between points with integer coordinates in $\mathbb{Z}^3$ will yield $\sqrt{integer}$ for any integer that is the sum of three squares: $a^2+b^2+c^2$. Interestingly, according to Lagrange's Four Squares Theorem[9] (see, e.g., [9], Theorem. 13.8 on page 500), every non-negative integer can be written as a sum of four integer squares: $a^2+b^2+c^2+d^2$. Therefore, considering lengths of line segments between points in $\mathbb{Z}^n$ for $n\geq 4$, every $\sqrt{integer}$ shows up!

6. Beginning in $\mathbb{Z}^n$ for $n\geq 3$ (which is beyond the traditional realm of constructible numbers) and allowing iterative constructions of hyper-planes (passing through previously constructed points)

[7] We point the interested reader to [6] for an explanation and exploration of sizes of infinity (i.e., cardinality) and to [3] for an exploration of the relationships between various number systems such as rational, algebraic, and complex numbers.

[8] For an enjoyable romp "on the flat", read E. Abbott's Flatland: A Romance of Many Dimensions [1].

[9] According to Rosen [9], Fermat had discovered the Four Squares Theorem, but had not published it; Euler tried, but wasn't able to prove it; Lagrange was the first to publish (1770) a proof.

and hyper-spheres (centered at and passing through a pair of previous constructed points), we get a dense cloud of points in n-space. However, the coordinates of these points are still nothing truly new – the coordinates are just real constructible numbers!

So, what does all of this mean? Although constructible numbers are countably infinite, an equally infinite (in size) subset of algebraic numbers are absent from the constructible numbers, since algebraic numbers contain roots of integral polynomials of all degrees whereas the constructible numbers only include roots of integral polynomials whose degrees are of a power of 2. Many (*in fact, most*) complex numbers are unreachable. Moreover, our extensions above reveal that going to higher dimensions does not produce anything new.

We finish by considering that our choice of tools greatly impacts what numbers (i.e., points) we can construct. In particular, our compass was assumed to be a *collapsing* compass. This means that if we pick it up, it does not stay fixed and open. If we used a non-collapsing compass, we could build in extra data like being able to transport fixed distances to new points. Likewise, our straightedge just allows us to draw lines. If we are allow to mark our straightedge, we would have some kind of ruler. If so, again we could transport distance information to new points. These modifications end up giving us access to more points than with a collapsing compass and an unmarked straightedge.

7 Student Investigations

We conclude this paper with some investigatory questions for students. These questions were curated for the student to glean more understanding of constructible numbers and ideas surrounding them.

1. What are (hyper-)planes and (hyper-)spheres?

   (a) Two points determine a line and three points a plane. (Usually anyway – when does this not work?) These are determined by equations of the form $ax + by = c$ and $ax + by + cz = d$. Explore why a hyper-plane in n-space is determined by $n + 1$ points. What kind of equation do we expect in general?

   (b) A hyper-sphere is the collection of all points equidistant (this is the radius) from a particular point (this is the center). What does this look like in 2-space? In 3-space? What kind of equations describe such shapes? Do all quadratic equations do so?

   (c) Starting with an n-dimensional grid ($\mathbb{Z}^n$) and intersecting hyper-planes determined by those points will just get us $\mathbb{Q}^n$. Allowing intersections of hyper-spheres and hyper-planes will still only allow us to construct points whose coordinates are real constructible numbers – we get essentially the same kind of stuff as we did in the plane. Explain this idea.

2. In the paper, we considered *points*, *lengths*, *areas*, and *volumes* as numbers. What others "*measures*" could be considered numbers? Explain why and how. (Hint) What if *angle* is a number?

3. Explain the following statements:

   (a) A real number x is constructible (via straightedge and compass) if and only if the field $\mathbb{Q}(x)$ over $\mathbb{Q}$ has degree a power of two, i.e., $[\mathbb{Q}(x) : \mathbb{Q}] = 2^m$ for some $m \geq 0$.

   (b) Any *angle* or *length* whose minimal polynomial over $\mathbb{Q}$ is of a degree that is not a power of two is not constructible.

   (c) A regular n-gon is constructible if and only if $n = 2^k \cdot p_1 \cdots p_m$ where p_i are distinct Fermat primes.

4. Examine Origami folding.

   (a) Examine the numbers that are constructible through Origami folding.

   (b) Did you find numbers different from or in addition to the constructible numbers? Explain.

5. Four ancient tools have connections to our discussions: the straightedge and compass, the framing square, and the Archimedean Spiral. Investigate all the lengths or angles constructible with a framing square and with an Archimedean spiral. Compare and contrast numbers constructed using these tools.

6. We could not construct $\sqrt[3]{2}$ using a compass and straightedge. However, investigate the "nuesis method" for constructing $\sqrt[3]{2}$. In particular, consider our Construct Cube Root of 2 worksheet linked in the footnote.[10]

7. Consider the table below. Write a historical term paper connecting, comparing, and contrasting the "Tools/Curves" listed in the following table and their respective "Typical New Constructible Numbers."

[10]This worksheet is accessible on the Maple cloud at https://maple.cloud/app/6240724865908736/Construct+Cube+Root+of+2?key=CA26EC FA8F8244CA9FE30DA34C43B6A378C94433A81E4710ACF0ECF789299515

Tool / Curve	Historical User	Algebraic or Transcendental Power	Typical New Constructible Numbers
Straightedge + Compass	Greeks	Quadratic only (2^n)	$\sqrt{r}$
Neusis / Framing Square	Greeks, Archytas	Cubic (3^m)	$\sqrt[3]{r}$
Archimedean Spiral / Conchoid / Cissoid / Trisectrix	Archimedes, Nicomedes, Diocles	Cubic	$\sqrt[3]{r}$, $\cos(20^\circ)$
Lemniscate or Quartic Algebraic Curve	Bernoulli, 17th century	Quartic (4)	$\sqrt[4]{r}$
Higher-Degree Algebraic Curves	Modern Algebraic geometry	Arbitrary Algebraic degree	Any Algebraic root
Quadratrix of Hippias	5th century BCE	Transcendental	Constructs π; squares the circle (in principle)
Logarithmic Spiral, Cycloid	17th century	Transcendental	Involves e, π, logarithms

8. Enjoy life, mathematics, and numbers.

References

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[5] E. Burger & R. Tubbs, *Making Transcendence Transparent*, Springer, New York, NY 2004.

[6] W. Cook and M. Bossé, "Types of Infinity", eJMT, Vol. 15 (2021) No. 2 (June).

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