

Computing Limits through Riemann Sums

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Abstract

It is well-known that Riemann Sums can be used to approximate certain definite integrals. In this paper, we will show how to use the idea of Riemann Sums to evaluate certain limits. It is done by expressing the desired limit as the limit of a suitable Riemann Sum. Several examples are given, as well as some exercises that are left for the reader to try on his own. An important aspect of the paper is that it encourages use of computer algebra systems such as Maple, at the same time emphasizing the importance of computation by hand.

In this paper, we will illustrate how to use the idea of Riemann Sums to evaluate certain difficult limits in calculus. The method we are about to discuss, even though not entirely new, is missing from most modern calculus texts (see [5]). It is worthwhile to recall the idea of Riemann Sums first. The Riemann Sums in general can be defined for a larger class of functions, but for simplicity we will just consider a monotonically increasing nonnegative function $f(x)$ defined on a closed interval $[0, a]$, where a is a positive real number. We know that the area bounded by the graphs of $y = f(x)$, $x = 0$, $x = a$, and $y = 0$ is given by the following definite integral:

$$Area = \int_0^a f(x) dx \quad (1)$$

Let us now divide the interval $[0, a]$ into n subintervals of equal length a/n , where n is any positive integer. One can construct two types of rectangles, using these subintervals as bases. The first kind has heights given by $f((i-1)a/n)$, where $i = 1, 2, \dots, n$. As given in Figure 1, the sum of the areas of these rectangles is called a Riemann Lower Sum of the function $f(x)$ over the interval $[0, a]$. Similarly, a second kind of rectangles have heights given by $f(ia/n)$, where $i = 1, 2, \dots, n$. As given in Figure 2, the sum of the areas of these rectangles is called a Riemann Upper Sum.

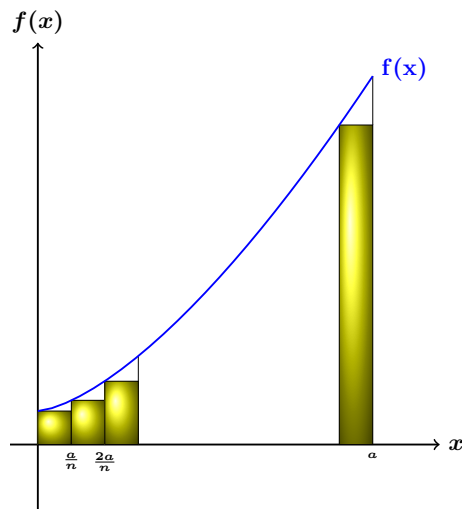


Figure 1: Riemann lower sums.

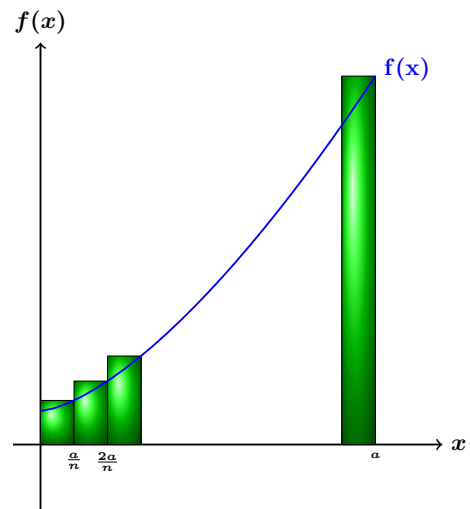


Figure 2: Riemann upper sums.

One central idea of integral calculus is that both the Riemann Lower Sum, and the Riemann Upper Sum are approximations for the area under the graph of $y = f(x)$ (see [3] and [4]):

$$Area = \int_0^a f(x) dx \approx \sum_{i=1}^n \frac{a}{n} f\left(\frac{ai}{n}\right) \quad \text{Riemann Upper Sum}$$

$$Area = \int_0^a f(x) dx \approx \sum_{i=1}^n \frac{a}{n} f\left(\frac{a(i-1)}{n}\right) \quad \text{Riemann Lower Sum}$$

For larger and larger n values, the above Riemann Sums will give better approximations for the area. More precisely, the limit of any of the Riemann Sums as $n \rightarrow \infty$ is equal to the true area under the graph.

$$Area = \int_0^a f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{a}{n} f\left(\frac{ai}{n}\right) \quad (2)$$

$$Area = \int_0^a f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{a}{n} f\left(\frac{a(i-1)}{n}\right) \quad (3)$$

The above Figures 1 and 2 only display nonnegative monotonically increasing functions. The same ideas can equally be used for nonnegative monotonically decreasing functions as well.

We will now show how to use the idea of Riemann Sums to calculate some interesting limits.

Example 1 Evaluate

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n+b} + \frac{1}{n+2b} + \cdots + \frac{1}{n+b(n-1)} \right), \text{ where } b \text{ is a positive real number.}$$

The idea is to rewrite the above limit as the limit of some Riemann Sum. First, observe the following:

$$\begin{aligned} \left(\frac{1}{n} + \frac{1}{n+b} + \frac{1}{n+2b} + \cdots + \frac{1}{n+b(n-1)} \right) &= \sum_{i=1}^n \frac{1}{n+b(i-1)} \\ &= \sum_{i=1}^n \frac{1}{n} \left(\frac{1}{1 + \frac{b(i-1)}{n}} \right) \\ &= \frac{1}{b} \sum_{i=1}^n \frac{b}{n} \left(\frac{1}{1 + \frac{b(i-1)}{n}} \right). \end{aligned}$$

Thus, our problem is equivalent to finding

$$\lim_{n \rightarrow \infty} \frac{1}{b} \sum_{i=1}^n \frac{b}{n} \left(\frac{1}{1 + \frac{b(i-1)}{n}} \right) = \frac{1}{b} \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{b}{n} \left(\frac{1}{1 + \frac{b(i-1)}{n}} \right)$$

By comparing with equation (3), one can see that the quantity

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{b}{n} \left(\frac{1}{1 + \frac{b(i-1)}{n}} \right)$$

represents the limit of the Riemann Lower Sum of the function $f(x) = \frac{1}{(1+x)}$ over the interval $[0, b]$. Thus it is equal to the definite integral $\int_0^b \frac{1}{1+x} dx$, which can be calculated by elementary calculus (see [3] and [4]):

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{b} \sum_{i=1}^n \frac{b}{n} \left(\frac{1}{1 + \frac{b(i-1)}{n}} \right) &= \frac{1}{b} \int_0^b \frac{1}{1+x} dx \\ &= \frac{\ln(1+b)}{b} \end{aligned}$$

Thus, the required limit is equal to $\ln(1+b)/b$.

Example 2 Evaluate

$$\lim_{n \rightarrow \infty} n^2 \left(\frac{1}{n^3 + 1^3} + \frac{1}{n^3 + 2^3} + \frac{1}{n^3 + 3^3} + \cdots + \frac{1}{n^3 + n^3} \right).$$

First note that

$$n^2 \left(\frac{1}{n^3 + 1^3} + \frac{1}{n^3 + 2^3} + \frac{1}{n^3 + 3^3} + \cdots + \frac{1}{n^3 + n^3} \right) = \sum_{i=1}^n \frac{1}{n} \frac{1}{\left[1 + \left(\frac{i}{n} \right)^3 \right]}.$$

Therefore, equation (2) implies that the required limit is equal to the definite integral of the function $f(x) = 1/(1 + x^3)$ over the interval $[0, 1]$, calculated as follows:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \frac{1}{\left[1 + \left(\frac{i}{n}\right)^3\right]} &= \int_0^1 \frac{1}{1 + x^3} dx \\ &= \frac{\ln 2}{3} + \frac{\sqrt{3} \pi}{9} \end{aligned}$$

The above integral can be either computed by hand, or by using a suitable computer algebra system (CAS). For example, the following command in Maple can be used to compute the required integral (see [1], [6] and [7]):

```
> int(1/(1+x^3), x = 0 .. 1)
```

Example 3 The reader is now encouraged to prove the following:

$$\begin{aligned} \lim_{n \rightarrow \infty} n^3 \left(\frac{1}{n^4 + 1^4} + \frac{1}{n^4 + 2^4} + \frac{1}{n^4 + 3^4} + \dots + \frac{1}{n^4 + n^4} \right) \\ = \frac{\sqrt{2}}{8} \ln(2 + \sqrt{2}) - \frac{\sqrt{2}}{8} \ln(2 - \sqrt{2}) + \frac{\sqrt{2} \pi}{8} \end{aligned}$$

Example 4 Calculate

$$\lim_{n \rightarrow \infty} \left(\frac{1^m + 2^m + 3^m + 4^m + \dots + n^m}{n^{m+1}} \right)$$

where, $m \neq -1$ is a real number.

Using methods similar to above, we can perform the calculation below:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{1^m + 2^m + 3^m + 4^m + \dots + n^m}{n^{m+1}} \right) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n} \right)^m \\ &= \int_0^1 x^m dx \\ &= \frac{1}{m + 1} \end{aligned}$$

Example 5 The reader is also encouraged to calculate the following limit:

$$\lim_{n \rightarrow \infty} \frac{1 + \left(\frac{4}{n}\right)^{1/n} + \left(\frac{27}{n^2}\right)^{1/n} + \left(\frac{256}{n^3}\right)^{1/n} + \dots + \left(\frac{m^n}{n^{n-1}}\right)}{n^{n+1/n}}$$

Example 6 Evaluate

$$\lim_{n \rightarrow \infty} \sqrt[n]{f(1/n) f(2/n) f(3/n) \dots f(n/n)}$$

where $f(x)$ is any function such that the integral $\int_0^1 \ln f(x) dx$ is convergent. Let

$$u = \sqrt[n]{f(1/n) f(2/n) f(3/n) \cdots f(n/n)}.$$

Taking \ln of both sides, we obtain

$$\ln u = \frac{1}{n} \left[\sum_{i=1}^n \ln f(i/n) \right].$$

Therefore, by methods described before, it follows that

$$\lim_{n \rightarrow \infty} \ln u = \int_0^1 \ln f(x) dx$$

This implies that the required limit is equal to

$$\lim_{n \rightarrow \infty} u = e^{\int_0^1 \ln f(x) dx}$$

We will record the result as follows:

$$\lim_{n \rightarrow \infty} \sqrt[n]{f(1/n) f(2/n) f(3/n) \cdots f(n/n)} = e^{\int_0^1 \ln f(x) dx} \tag{4}$$

Example 7 Using the above Example 6, the reader can show the following identities:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{e^{1/n} \cdot e^{2/n} \cdot e^{3/n} \cdots e^{n/n}} &= \sqrt{e}, \\ \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{\left(\frac{1}{n}\right)^m \cdot \left(\frac{2}{n}\right)^m \cdot \left(\frac{3}{n}\right)^m \cdots \left(\frac{n}{n}\right)^m}} &= \sqrt[n]{e}, \\ \lim_{n \rightarrow \infty} \frac{2n}{(2n)!!} &= e, \\ \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{\left(\frac{1}{n}\right)^{1/n} \cdot \left(\frac{2}{n}\right)^{2/n} \cdot \left(\frac{3}{n}\right)^{3/n} \cdots \left(\frac{n}{n}\right)^{n/n}}} &= \sqrt[4]{e}, \\ \lim_{n \rightarrow \infty} \frac{1}{\sqrt[2n]{\left(\frac{1}{n}\right) \cdot \left(\frac{2}{n}\right) \cdot \left(\frac{3}{n}\right) \cdots \left(\frac{n}{n}\right)}} &= \sqrt{e}, \\ \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{\sin\left(\frac{\pi}{2n}\right) \cdot \sin\left(\frac{2\pi}{2n}\right) \cdot \sin\left(\frac{3\pi}{2n}\right) \cdots \sin\left(\frac{n\pi}{2n}\right)}} &= 2. \end{aligned}$$

Example 8 Evaluate

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n}$$

We will compute this limit using equation (4) with $f(x) = x$:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} &= \lim_{n \rightarrow \infty} \frac{\sqrt[n]{1 \cdot 2 \cdot 3 \cdot 4 \cdots n}}{n} \\ &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n}{n}} \\ &= e^{\int_0^1 \ln x} \\ &= \frac{1}{e} \end{aligned}$$

The limit in this example can also be calculated using the following command in Maple:

```
>limit(factorial(n)^(1/n)/n, n = infinity)
```

Though Maple uses sophisticated algorithms for finding limits, we cannot expect Maple to calculate any type of limit. In an attempt to calculate the limit

$$\lim_{n \rightarrow \infty} \left(\frac{1^m + 2^m + 3^m + 4^m + \cdots + n^m}{n^{m+1}} \right),$$

in Example 4, the following Maple command does not produce an answer:

```
>limit((sum(k^m, k = 1 .. n))/n^(m+1), n = infinity)
```

In this paper, we have shown a method of calculating certain limits via suitable Riemann Sums. In some cases we were able to perform the calculation by hand, but in other cases, the calculation was facilitated by a CAS. However, we hope that the paper shed some light into the limitations of a CAS as illustrated by Example 4.

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