Building LVAR (Linking Visual Active Representations) modes in a DGS environment

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Abstract: *The present study presents the different LVAR modes which can be constructed in Geometer's Sketchpad v4 dynamic geometry software. The paper posits an explanation of the correlation between the five phases in the apprenticeship/learning process proposed by van Hiele and the developing theory on LVAR. A few examples of the different modes of LVAR are presented, including the answers of the student participants in the didactic experiment conducted. We can thus conclude that transformations through LVAR lead students to structure mental transformations relative to the development of their van Hiele level.*

1. Introduction

The paper is about secondary school students' development of geometrical reasoning in a dynamic geometry environment, and focuses on the interaction between the students and the different modes of linking representations facilitated by Geometer's Sketchpad v4[36] dynamic geometry software. The paper touches on a research area which is still under debate in the mathematics education community: the effect that external representations (for example, software tools) and human interaction (for example, teacher guidance or classroom discourse) can have on students' cognitive development, considering both social and cognitive dimensions in the study of the problem-solving process in the dynamic geometry context. Some questions the researcher had in mind when conducting her research and when designing the problems in the DGS environment were the following: How does the idea of Euclidean proof correlate with the van Hiele model of geometrical thinking? How can we transfer this idea into the dynamic geometry environment, exploiting the role that dynamic representations play in the development of students' geometrical reasoning?

Concretely: During a didactic experiment conducted in Greece with the support of Geometer's Sketchpad v4 dynamic geometry software, student participants followed a 4-phase DG research-based curriculum conceived and employed by the researcher as part of her PhD thesis, which is still in progress. She was responsible for the choice of activities, for session planning and for student assessment. The DG curriculum was composed of four phases: Phase 1 – *construction* activities; Phase 2 – *construction through symmetry* activities; Phase 3 – *the exploration of open-ended problems*; and Phase 4 – building and transforming semi-predesigned *Linking Visual Active Representations (LVAR)* [55]. It was to further explore phase 4 that the researcher conducted a didactic experiment with (semi) pre-designed multiple-page sketches detailing the sequential phases of the solution to the problem under investigation using rigorous proof. In so doing, she transferred her classroom teaching style into the software problem-design, drawing on Socrates' chain-questioning method, which aims to stimulate interaction. For this reason, she linked all the software

actions using the interaction techniques supported by the Geometer's Sketchpad v4 (DGS) environment to better allow students to discover solution paths and to reason by rigorous proof.
This mode of design and the results of the experimental use of the software with students led to the need to define two new concepts: Linking Visual Active Representations (LVAR) and Reflective Visual Reaction (RVR).

Firstly, the terms that have been chosen by the researcher to define the concept of LVAR require illustration.

- The term '*linking*' was preferred to 'linked' because the former denotes something that can be linked, but is not necessarily linked at this moment. It will be explained in greater detail why the diagrams are only partially pre-constructed in the examples of the problems and illustrations in Section 6.
- All DGS objects are necessarily '*visual*' representations of what they stand for.
- An '*active*' representation is a representation that causes action, motion or change because it is in operation, in effect or in progress. Dynamic representations can always *become active* if we cause an action on them, but they are not always pre-constructed. LVAR *always involve semi pre-constructed* dynamic diagrams that can be linked and become active in accordance with the wishes of the user, meaning the user is not limited to "actions pre-set by the sketch creator" [65].

Linking Visual Active Representations and Reflective Visual Reaction during a dynamic geometry problem solving session are defined as follows [55], [56]:

Linking Visual Active Representations are the successive phases of the dynamic representations of a problem which link together the problem's constructional, transformed representational steps in order to reveal an ever increasing constructive complexity. Since the representations build on what has come before, each one is more complex, and more integrated than the previous ones, due to the student's (or teacher's, in a semi-preconstructed activity) choice of interaction techniques during the problem-solving process, aiming to externalize the transformational steps they have visualized mentally (or existing in their mind).

Reflective Visual Reaction is the reaction based on a reflective mode of thought, derived from interaction with LVAR in the software, thus complementing and adding to the student's pre-existing knowledge or facilitating comprehension and integration of new mathematical meanings.

The results of the research can be illustrated as follows [55]: LVAR motivated the students to answer rapidly and spontaneously. The researcher kept her questions coming fast, which meant students did not have time to use paper and pencil. The classroom observations revealed that the same students did not always display the same spontaneous reflex reactions. The LVAR that spread over multiple pages helped the students to react instantaneously and to articulate their thoughts. The LVAR helped the students to operate in an auxiliary or complementary manner, assimilating or accommodating their prior knowledge, or as a confirmation of the student's cognitive processes. The students' RVR occurred at many points during the didactic experiment thanks to the use of interaction techniques. As a result the students constructed mental schemes for mathematical meanings and were "starting to develop longer sequences of statements and beginning to understand the significance of deduction" [15]. LVAR helped the students form rigorous Euclidean proofs and they reached conclusions on the problem by correlating the theorems they already know. This is to say that LVAR assisted students to develop their van Hiele level.

In the present study, the different modes of LVAR are presented correlated with the phases developed by Dina van Hiele-Geldolf.

2. The van Hiele model and the five phases of apprenticeship/learning process

Pierre van Hiele and his wife Dina van Hiele–Geldof developed a theoretical model involving five discrete levels of thought development in geometry ([22] p.6). According to Dina van Hiele-Geldof ([22] p.16) the didactic experiments she discussed sought "to investigate the improvement of learning performance by a change in the learning method". Her study investigated whether "it was possible to use didactics as a way of presenting material, so that the visual thinking of a child is developed into abstract thinking in a continuous process, something that is requisite for logical thinking in geometry". The study focus was influenced by concerns formulated by Crowley [13], who argues that "the need is for classroom teachers and researchers to refine the phases of learning, develop van Hiele-based materials and implement those materials and philosophies in the classroom setting, (so that) geometric thinking can be [made] accessible to everyone". Many researchers (Burger & Shaughnessy [10], for instance) have argued that sequencing instruction has positive effects on students' success. Burger & Shaughnessy claim that if initial activities are not interesting or are too easy, they might not attract or motivate students to focus on the topic and might not bring with it a sense of success. The five levels of thinking reflect on students' progress and increasing development in the way in which they are able to reason about geometrical objects and their relationships, and focus "on the role of instruction in teaching geometry and the role of instruction in helping students move from one level to the next" ([22] p.6).

Central to this model, is the description of the five levels (see for example [15], [24] p.361):

- Level 1 (recognition or visualization): students visually recognize figures by their global appearance. The properties of the figure are not explicitly identified or perceived.
- Level 2 (analysis): students start analysing the properties of figures and learn the appropriate terminology for their description. They can recognize and name properties of geometric figures, but do not see relationships between these properties.
- Level 3 (ordering): students logically order figures' properties by short chains of deductions and understand the interrelationships between figures.
- Level 4 (deduction): students start developing longer sequences of statements and begin to understand the significance of deduction, the role of axioms, theorems and proof.
- Level 5 (rigor): Students at this level understand the formal aspects of deduction, such as establishing and comparing mathematical systems.

Another important aspect of the van Hiele model is the five phases it specifies in the apprenticeship process. This model of teaching phases, as discussed below, is used as the main theoretical framework for the interpretation of the LVAR modes in this paper. Instruction that takes this sequence into account promotes the acquisition of a higher level of thought. The five phrases are described below (see [22] p.251; [71]):

- Phase 1 (Information): Through discussion, the teacher identifies what students already know about a topic and the students become oriented to the new topic.
- Phase 2 (Guided orientation): Students explore the objects of instruction in carefully structured tasks such as folding, measuring, or constructing. The teacher ensures that students explore specific concepts.
- Phase 3 (Explicitation): Students describe what they have learned about the topic in their own words. The teacher introduces relevant mathematical terms.
- Phase 4 (Free orientation): Students apply the relationships they are learning to solve problems and investigate more open-ended tasks.

- Phase 5 (Integration): Students summarize and integrate what they have learned, developing a new network of objects and relations.

3. Related research studies

Battista [7] found that spatial visualization and logical reasoning were important determinants of geometry achievement. Clements, Battista, and Sarama [12] found that a Logo-based curriculum helped elementary students perform better in a range of geometrical tasks, while Sedig, Klawe, and Westrom ([63] quoted in [64] p.47) found that "adding scaffolding to direct manipulation of representations of transformation geometry concepts significantly improved student learning". Dynamic geometry systems such as the Geometer's Sketchpad [36] or Cabri II [44], (or any other DGS software) are microworlds designed to facilitate the teaching and learning of Euclidean geometry. In Gawlick's opinion ([24] p.370) a *dynamic* approach is more appropriate to developing advanced level thinking both because tasks prepared for lower levels can be continued to higher levels, thus accustoming students to the habit of 'discovery', and because it provides a material base for the sequential van Hiele *phases of learning* since the students can *explore* the topic in a directed orientation phase and then *build the new concepts* for themselves, drawing on their *previous knowledge*.

The Geometer's Sketchpad is a highly visual dynamic tool for exploring and discovering geometric properties. Many researchers who used the Sketchpad have conducted studies, using the van Hiele model as descriptor for their analysis and concluded that students displayed more positive reactions when testing conjectures and constructions [28] and achieved significantly higher scores on a test containing concepts ([1, 18, 33, 70].

4. The role of LVAR in theoretical thinking

Researchers around the world agree that learning is a complex process, being both constructivist as it depends on active individual construction, and sociological, since it becomes part of a culture by dint of having socio-cultural aspects. The general framework owes much to Piaget's approach: faced with a sufficiently problematic context, the learner has to negotiate gaps in or inconsistency problems with his/her knowledge. As the students become familiar with the technological tools, they control their world, and their cultures and modes of knowing thanks to their acquired competence [9]. Since tools exert an influence over the technical and social way in which students conduct an activity, they are considered essential to students' growth and development.

Dynamic geometry software packages are *representational infrastructures* [40] that may be used to make changes both in geometry and the expression of geometric relationships, and hence in the teaching and learning of mathematical concepts. These systems can play an intermediary/mediatory role in organizing students' thought processes, allowing them to construct an internal representation based on an external model [38]. In Geometer's Sketchpad v4 DGS environment, LVAR are interpreted as "encoding the properties and relationships for a represented world consisting of mathematical structures or concepts" ([64], p.2 in line with Goldin and Janvier [27] p.1): "a physical situation, or situation in the physical environment", modeled mathematically embodying mathematical ideas; a combination of "syntactic and structural characteristics" enhanced by selected different interaction techniques facilitated by the Geometer's Sketchpad v4 DGS environment where the problem is transferred or a geometrical theory is discussed; a formal mathematical proof, "usually obeying axioms or theorems or conforming to precise definitions, — including mathematical constructs that may represent aspects of other mathematical constructs"; "an internal, individual cognitive configuration, inferred from behaviour or introspection, describing some aspects of the processes of mathematical thinking and problem solving".

The goals in developing LVAR were to

1) Provide DGS–based problems that are adaptations and extensions of existing activities,

2) Get students to solve problems individually or in a classroom orchestrated process, which develops mathematical understanding and formal mathematical proof

3) Provide experiences that are more effectively presented by selected interaction techniques facilitated by the DGS environment than by other didactic materials and

4) Provide these experiences in the context of the figurative or drawing design mode, by means of which students develop their aesthetic sense and acquire actual cognitions in geometry. This latter idea is in accordance with that of Parzysz [54].

Parzysz defined a drawing as a representation of a geometrical object and a figure as the "text defining it [the geometrical object]". Hollebrands [33] writes that "building on Parzysz's ideas, Laborde defines drawing as that which refers to the material entity (the physical drawing) while figure refers to the set of discursive representations and diagrams referring to the geometrical referent (the theoretical object)". LVAR in the Dynamic Geometry environment play a complementary role exactly as Laborde [43] reports for the diagrams in the plane geometry: "on the one hand, they refer to theoretical geometrical properties, while on the other, they offer *spatiographical* properties that can give rise to a student's perceptual activity". In the same way, LVAR exactly link the material digital entities on the screen with the theoretical mental referent which can be worked on. This is to say that LVAR simultaneously build on the duality of diagrams, being both representations of concepts and theorems or mental constructs and representations of a combination of active geometrical objects. Thus, when one acts on LVAR, conducting transformational operations on mathematical objects and reasoning on the basis of the conceptual properties of geometric figures, the theoretical part is shaped as a mental entity that simultaneously includes *spatiographical* recognitions and properties.

Sedig and Sumner [64] have distinguished between *basic* and *task-based* interactions with visual mathematical representations (see Figure 1). To achieve student interaction using LVAR, the researcher used a diverse set of interaction techniques including "*animating*" a point on its path, '*tracing*" a segment, "*hiding and showing*" action buttons, and "*linking*" or "*presenting*" action buttons. In so doing, the researcher successfully linked both the steps in constructional and transformational actions and the various sequential phases in the proof. According to Lagrange [45] "a technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency."

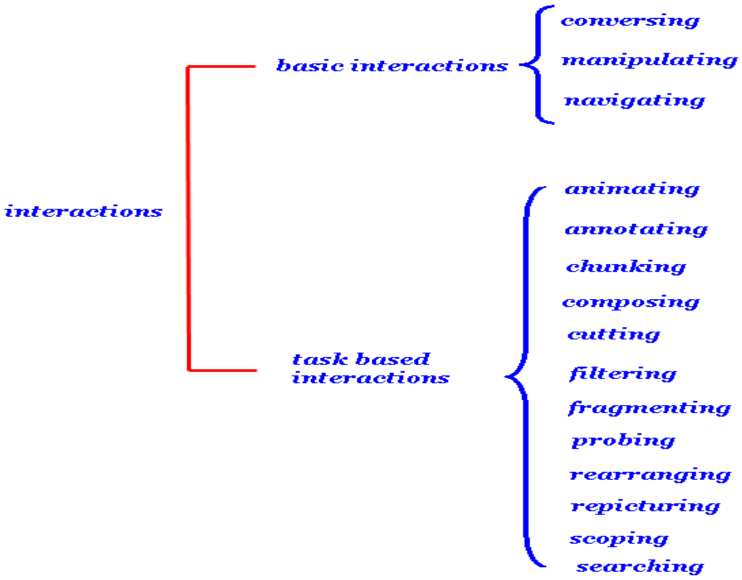


Figure 1: Basic and task-based interactions defined by Sedig &Sumner [64]

Through LVAR, the teacher can guide the students by means of elucidation or questions eliciting conclusions which form a step-by-step visual proof. The successive pages in the software also play a significant role, and can be seen as a vivid section in a book revealing the various stages in the proof. The sequence of increasingly sophisticated construction steps could thus correspond to the numbering of the action buttons which allows student to interact with the tool when they want to or when they are encouraged to do so by their teacher in class.

The theoretical framework includes the notions of instrumental genesis [69] and the distinction between phases of instrumentation and instrumentalization (see for instance [2, 3, 4, 29], which are fundamental to teaching in computer environments. During the instrumental genesis, both phases (instrumentation and instrumentalization) coexist and interact. The user then structures what Rabardel [58] has called the utilization schemes of the tool/artefact.

Utilization schemes are the mental schemes that organize the activity though the tool/artefact. This process has been addressed in many studies, based on the research of Verillon and Rabardel [69] into the means by which an artefact becomes an instrument for a student.

As Trouche sees it, "instrumental geneses are individual processes, developing inside and outside classrooms, but including of course social aspects" (personal e-mail correspondence with Luc Trouche on April 4, 2008 quoted in [55], see Figure 2). Trouche argues that "an artefact is transformed thus through instrumental geneses, oriented by finalized actions, assisted by instrumental orchestrations, into an instrument".

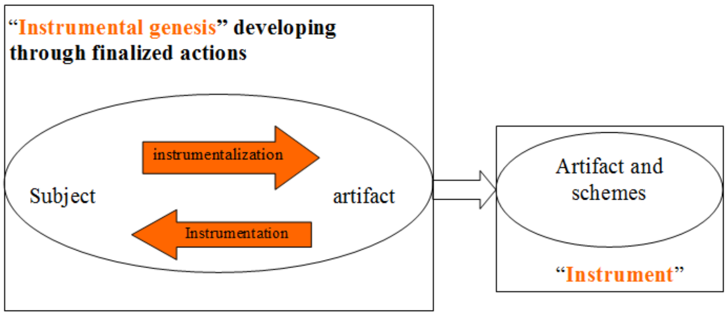


Figure 2: The schema of instrumental approach (personal e-mail correspondence with Pr. Luc Trouche on April 2, 2008, based on Trouche's [68] instrumental approach, quoted in [55])

Artigue [4] supports that "an instrument is thus seen as a mixed entity, constituted on the one hand of an artefact and, on the other hand, of the schemes that make it an instrument for a specific person. These schemes result from personal constructions but also from the appropriation of socially pre-existing schemes." Artigue [3] reports on the 'genesis of reflection about instrumentation issues, and the dialectics between conceptual and technical work in mathematics'. Reflecting on dynamic diagrams involved consciously representing the actions or mental processes and then considering their results or composition. The students act on dynamic diagrams (for instance, LVAR) to construct their knowledge or to investigate the problem solution, and interact with the dynamic diagrams to express their thoughts. In this case, they use dynamic diagrams as tools/ artefacts with which to shape their thoughts. Noss and Hoyles [48] argue that while student activity during instrumental genesis in a computing environment is shaped by their tools (in our case, dynamic LVAR); the students in turn shape the dynamic LVAR in order to express their arguments. During the construction of a diagram (or an action upon a diagram) the student structures an internal invisible side of the representation which is part of the process being applied to the external representation.

LVAR give the users-students the opportunity to improve/facilitate their understanding and to move to a higher van Hiele level by acting in an auxiliary or complementary manner, assimilating or accommodating students' prior knowledge, or confirming their thought processes/mental approach. In the words of Kaput [38, 39] "a representational framework for mathematical cognition and learning is consistent with constructivism". During the interaction with LVAR, students interplay with the spatiographical features of the diagrams and their spatial characteristics and construct a deep understanding of their properties referring to the theoretical object from the "reflection shaped by the tools and the language operationalized by them " [49].

This means that through LVAR and the operationalization of reflective abstraction, previously formed or structured abstract items of mental operations can become the content in future acts of abstraction. According to Hollebrands (ibid.) "students are usually asked to work on material drawings, the spatial graphical features of the signifier, but they are expected to reason about figures, the signified....To reason more formally about geometrical properties, rather than just about the spatial characteristics of diagrams, students need to have deep understandings of those properties. Deeper understandings may be indicative of a student who engages in reflective abstraction and possesses an object conception of a concept ([33], p.59)".

5. Semi-preconstructed Diagrams and the Proving process

… "Students cannot begin to do a question about geometric objects because they can't construct the diagram, and they can't construct the diagram because they don't understand the connections between geometric objects…" (Sinclair, M., [65] p.5)

Mathematical reasoning is based on the conceptualization of the "if-then" statements. Van Hiele Level 4 (deduction) is identified as the level which is connected with the construction of the "if...then" statements and consequently with proof [24]. Research has shown that proof is difficult for most students (see for example [34, 11, 35, and 30]). The challenge is therefore to design situations that are motivating and that help strengthen reasoning and argumentation skills [65]. In support of this, Gawlick ([24], p.362) declares that "progression through the (van Hiele) levels will not occur all by itself, but needs to be triggered by giving the students suitable tasks that really afford the building of new concepts".

Many researchers have investigated the role of proof teaching and learning in school, tackling it from different perspectives. Specifically, they have analyzed the cognitive processes involved in the construction of proofs (see for example [19], [30]) or the role of the teacher with regard to the proof process (see for example [5]). Another branch of the research concerns the impact of computer technologies on the class [62]; more specifically, the impact of dynamic geometry software on the proving process (see for example [26, 14, 42, 51, 52]. Edwards [20] argues that "to effectively support the teaching of proof with meaning, we must understand *how* students learn to reason […] and *how* these processes can be supported in the learner". Research has shown that even when working with static means, students start conjecturing when faced with a proving process in a problem situation. Depending on the tools with which they are provided and their interaction with the teacher or other students, they can develop elements of deductive reasoning by "developing specific competencies inherent in producing conjectures and proving the produced conjectures by taking elements of theoretical knowledge into account" [8]. Although the two phases—conjecture production and proof construction—cannot be separated and linearly sequenced, their component elements are described and reported by Boero [8]: (1) producing a conjecture (which includes exploring the problem situation, identifying possible "regularities" and the conditions under which such regularities take place, identifying arguments for the plausibility of the produced conjecture); (2) formulating the statement according to shared textual conventions; exploring the content of the conjecture and the limits of its validity (which includes heuristic, semantic (and even formal) elaborations about the links between hypothesis and thesis, identifying appropriate arguments for validation related to the reference theory, and envisaging possible links amongst them); (3) selecting and enchaining coherent, theoretical arguments into a deductive chain, frequently under the guidance of analogy or appropriate, specific cases; (4) organizing the enchained arguments into a proof that is acceptable according to current mathematical standards; and (5) approaching a formal proof (or parts of the proof).

Heinze [31] modified this sequence into five coding categories, of which the last three categories are: Phase 3 – an explorative phase based on the formulated conjecture and aimed at identifying appropriate arguments for the conjecture and a rough planning of a proof strategy, which can be divided into four subcategories: (1) referencing assumptions, (2) investigating assumptions, (3) collecting further information and (4) generating a proof idea; Phase 4 – the combination (verbal or written) of these arguments into a deductive chain that constitutes a sketch of the final proof; and Phase 5 – the writing down of the chain of arguments according to the standards of the mathematics classroom in question (including a retrospective overview of the proof process).

Barwise and Etchemendy [6] analyzed the role of diagrams in the proof and proving process, concluding that since diagrams, like sentences, carry information, carving up the same space of possibilities, albeit in very different ways, they can play an integral role in reasoning and constitute legitimate elements in mathematical proofs (Barwise and Etchemendy, see [6], p.9.) From Peirce' s theoretical perspective, diagrams "are the products of mental imagery, subordinated to mental transformations and reinterpretations, and serving as vehicles for conveying the meanings ascribed to them by the individual" ([59], p.855). Peirce [57] indicates, "a diagram is an icon of a set of rationally related objects … (which) not only represents the related correlates, but also and much more definitely represents the relations between them…". Herbst [32] suggests four possible modes in which students' interaction with a diagram, and their discourse about the geometric objects at stake in the diagram, may relate. These modes of interaction (empirical, representational, descriptive, and generative) instantiates a set of relationships between the subject, the diagram and the theoretical, geometric object at stake in that interaction. Diezmann [16, 17] argues that diagrams have three key cognitive advantages in problem solving: they facilitate the conceptualisation of the problem structure, which is a critical step towards a successful solution [21]; they are an inference-making knowledge representation system [46] that has the capacity for knowledge generation [41], and they support visual reasoning, which is complementary to, but differs from, linguistic reasoning [6]. According to the van Hiele model, the role played by diagrams in justification varies depending on the level. Lower-level students approach the diagram globally, forming arguments and constructing informal proofs on the basis of what the diagram includes. Higher-level students are more familiar with a diagram's properties, and understand that it represents abstract geometric concepts. Geometric proofs require students to transcend what they can see and relate diagrams to geometric concepts they already know: as such, students must be helped to develop their diagram interpretation abilities.

Laborde posits that dynamic geometry software diagrams provide for a new kind of diagram "….when the user drags one element of the diagram, it is modified according to the geometry of its constructions rather than the wishes of the user" ([43] p.165). Laborde concludes that "Dynamic Geometry environments break down the traditional separation between action (as manipulation associated with observation and description) and deduction (as intellectual activity detached from specific objects) and reinforce the moves between the spatial and the theoretical domains". In support of this, Smith and Hollebrands [66] write that dynamic geometry software programs "enable students to construct accurate diagrams and interact with the diagrams to abstract general properties and relationships, because the ways in which the programs respond to the students' actions is determined by geometrical theorems".

Sinclair ([65], p.136) observed secondary school students in a dynamic-geometry supported classroom, examining the interrelationships between the students and the elements of the learning task in order to describe the benefits and limitations of pre-constructed diagrams in Java Sketchpad [37] with regard to the development of reasoning skills related to geometric proof. She concluded ([65] p.136): "The study results show that Java Sketchpad motivates and engages students. It helps students strengthen their geometric thinking skills-especially at the visualisation and analysis levels, by supporting student exploration, visual reasoning, and communication activities. …Through my analysis I hope that I have furthered our understanding of the role that pre-constructed dynamic sketches can play in a geometry learning situation. And I hope that my analysis will help teachers and researchers to…design tasks that are not only do-able, but also worth doing".

In light of this point of view, the researcher hopes to contribute to the research being conducted into the benefits of using semi-preconstructed diagrams in a geometry proving process which, facilitating manipulation and exploration, are also *active* since they allow students to act on and modify them using the full range of program features. Specifically, the process of proving a problem or theorem consists of a series of steps which can function as responses anticipating the questions posed explicitly or implicitly by teacher or student. This is what the researcher had in mind when she designed the different semi-preconstructed LVAR modes in Sketchpad (to be reported in the next section) to link the proving process with envisaging "arguments for the plausibility of the produced conjecture, appropriate arguments for validation related to the reference theory" and combining "these arguments into a deductive chain that constitutes a sketch of the final proof" ([8, 31]. The research has led her to conclude that the semi-preconstructed LVAR diagrams have the following features:

- They help "make the final configuration less complex because all the inevitable auxiliary intermediate lines that must be drawn to achieve the final construction" [61] do not appear immediately.
- They appear in stages in dynamic linking illustrations, which help to keep the students focused on the aim of the overall construction.
- "They are valuable as learning tools enhancing the ability to recognise the connections between geometric objects" [65]; and
- They can be acted on and modified by students, allowing them to use the full range of program features (which renders them ***Active***).

Having observed that there are several ways of characterizing different connections/links between representations correlated using the different interaction techniques supported by the Sketchpad, the researcher tried to produce a characterization for this different mode of linking representations. Consequently, the main question concerns the way in which the interaction relates to the different LVAR modes in the software through the development of conjectures, meaning statements that could lead to (or be underlain by) an inductive or deductive mode of thought or proof. The next section presents the different LVAR modes. Screenshots of the sequential representations of two problems modeled in the software are presented and correlated with excerpts from dialogues recorded during the research process in which I have identified students' arguments or conjectures and students deductive chains to construct their solutions to the problems presented to them.

Examples 1 and 2 are parts of the solutions of Problems 1 and 2 representing the different LVAR modes. The first problem is a revision of the problem created by George Gamow [23] involving pirates and buried treasure. Gamow's problem hinges on a treasure map found in an old man's attic. Here is the revision provided by the researcher [55, 56]: "In the Odyssey, Homer (c74-77) mentions that the pirates also raided Greek islands. The pirate in our story has buried his treasure on the Greek island of Thasos and noted its location on an old parchment: "You walk directly from the flag (point F) to the palm tree (point P), counting your paces as you walk. Then turn a quarter of a circle to the right and go to the same number of paces. When you reach the end, put a stick in the ground (point K). Return to the flag and walk directly to the oak tree (point O), again counting your paces and turning a quarter of a circle to the left and going the same number of paces. Put another stick in the ground (point L). The treasure is buried in the middle of the distance of the two sticks (point T)." After some years the flag was destroyed and the treasure could not be found through the location of the flag. Can you find the treasure now or is it impossible?" The problem has attracted many researchers. For example, Scher [60] provided a solution to the problem in multiple pages making interactive constructions by using the Geometer's Sketchpad.

The second problem is the following: "A power plant is to be built to serve the needs of the cities of A (Athens), B (Patras) and C (Thessaloniki).Where should the power plant be located in order to use the least amount of high-voltage cable that will feed electricity to the three cities?"[50]. The researcher carried out the didactic experiment bearing in mind Dina Van Hiele-Geldof's didactic approach ([22], p.185), which promotes the building up of geometry through structure, directs students' thinking activity to an analysis of structure prior to the formation of associations, and simultaneously provides an opportunity for the student to develop structuring-focused thinking.

The didactic experiment was conducted in a class at a public high school in Athens, Greece, during the second term of the academic year, and involved 14 students aged 15-16 who made up the experimental team. The methodology used here includes case studies of pairs of students. The data were analyzed using grounded theory's constant comparative method.

6. What are the different modes of the LVAR? What is the relation between LVAR and the phases of the van Hiele model?

6.1 Mode A-the inquiry/information mode

A part of the problem 1 (or 2) or a problem under investigation /solution requires the use of an action button (animation, for example, or the trace command) to render the result visible during the investigation stage. The original diagram is transformed into a "diagram in motion", reinforcing the original image since the stimulus received from the visual representation leaves the properties of the figure unaltered despite the transformation it undergoes. The students explore the problem and start *using task–based interactions* like *animating and temporary annotating (tracing) as well as basic interactions* like *dragging.*

In this *mode of LVAR the* students familiarize themselves with the field under investigation using the instantiated parts of the diagrams which lead them to discover a certain structure through their interaction with the diagrams or during discussions. "Reflection upon the manipulation of material objects, by taking the relations between those shapes as an object of study, can lead to geometry" (Dina van Hiele in [22]).

Example from the first problem: When the students interact with the linking visual representations, they can visualize the sequential steps of all the visual representations that appear during the animation of point F as the segment KL is traced. The students began by experimenting with the position of F on segment PO (see [55] p.370). In this way, the researcher constructed the first mode of positioning the point on the segment. The students can verify visually that the distances KT, TL remain equal as point F is moved along PO and that T remains the midpoint of KL for every point F. Namely, the depicted representations display spatial-graphical shapes and their relations. The segment KL leaves traces on screen which shapes/forms a quadrilateral with specific properties. The students recognize the shape of the square in it and reach conclusions on its properties from the diagram (see Figure 3).

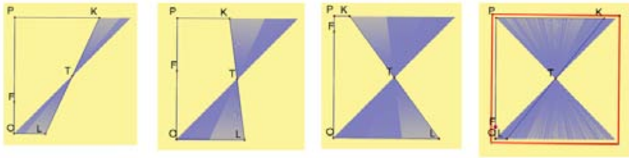
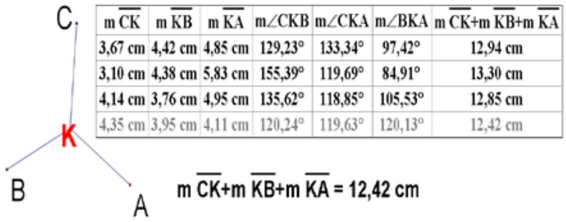


Figure 3: Example from the first problem - Sequential phases of the figure while point F is animated and KL is traced



m $\overline{CK}$	m $\overline{KB}$	m $\overline{KA}$	m∠CKB	m∠CKA	m∠BKA	m $\overline{CK}$+m $\overline{KB}$+m $\overline{KA}$
3,67 cm	4,42 cm	4,85 cm	129,23°	133,34°	97,42°	12,94 cm
3,10 cm	4,38 cm	5,83 cm	155,39°	119,69°	84,91°	13,30 cm
4,14 cm	3,76 cm	4,95 cm	135,62°	118,85°	105,53°	12,85 cm
4,35 cm	3,95 cm	4,11 cm	120,24°	119,63°	120,13°	12,42 cm

Figure 4: Example from the second problem - Sequential phases of experimentations with the lengths of the segments and the angles

This process results in a connection between the ‘spatiographical’ and the ‘theoretical field’ as Laborde [43] describes. The students react to the visual stimulus and respond instantaneously. Their response is a result of the reaction which occurs as a result of the visual stimulus. This is to say that the students mentally transform the meaning of the congruence/equality of the segments perceived visually in the diagram into the meaning of symmetry. This means that the students construct an instrument out of their interaction with the tool which also includes an instrumented action scheme relating to the meaning of symmetry. The depicted representations lead the students to recognize a more sophisticated representation globally. At the same time, this process results in the students visually connecting the meaning of the traced square (see Figure 3) both with the equality of its sides and with the equality of its diagonals—i.e. a relationship between the two meanings.

Here is a discussion between the researcher and the students [55]:

218. Researcher: Which is the position of point T as we drag point F? *(RVR)*
219. Student M_4: it is the ***symmetry centre*** *of the shape.*
220. R: Can you conjecture what kind of quadrilateral is being shaped?
221. All the students: it looks like ***a square.*** *(RVR)*

Example from the second problem: The students investigate the modifications made to the calculations of the segments to identify the different different positions of point K. Changing the position of point K by dragging it is dynamically linked to the changes/ modifications in the resultant angles in the table and the upcoming modification to the sum of the segments. This process encourages students to observe that the minimal sum is observed when the angles are at 120^o (Figure 4).

The students are usually led to draw rough conclusions regarding the position of the point under investigation; for instance, that it is the circumcentre of the triangle ABC. The construction of the circumcentre and the measurements reveal cognitive conflicts in the students. The addition of a new line in the table for new measurements every time point K is dragged can lead students to posit conclusions which converge on the angles between the segments being 120 degrees. During this process, we have a reversible (bi-directional) transformation of a) the geometrical into an algebraic model, and b) the algebraic conclusions drawn from comparisons between on-screen dragging on the geometrical representation.

As Dina van Hiele explains [22], students are able to perceive structure in almost anything, however unordered. Because different students can perceive this structure in the same way, they can discover the intrinsic ordering in the material presented; for example, knowledge of shapes is developed through the manipulation of material objects.

6.2 Mode B: the directed orientation mode

The sequential phases of the problem under investigation are displayed as a global shape to which more elements and/or information are gradually added when action buttons are pressed. The steps in the construction of the diagrammatic reconstruction which are displayed by pressing the action buttons are linked to suitable questions and their answers.

According to Olivero ([53], p.279) "The possibility of hiding and showing elements... is a powerful tool of dynamic geometry software, because according to what is left visible the focus can shift to different elements. Hiding or showing elements of a configuration at stake changes the nature of the figure to explore because what are visible changes and therefore the potential elements of the focusing process change too. What students see on the screen influences the construction of conjectures and proofs and choosing what they want to see on the screen influences the proving process". The definition of hide-show action buttons allowed the students to develop their '*direct manipulation*' of the diagram on the screen using a *basic interaction*. The students explored the problem and started *using task–based interactions* like *filtering*, *rearranging, annotating, and probing* parts of the figure.

In concrete terms, the sequential constructional steps of the solution to the problem emerge step by step. The process has the following advantages: the student can recall/redisplay the correct answer to his question or the teacher's question which appears when he clicks on the appropriate button; the process can be repeated as many times as the student wants, which saves time in a proving process.

The students discover an important part of the solution to the problem on the same page of the software by means of the gradual display of increasingly complex questions which are connected to the revealing/concealing of parts of the configuration of the problem, and which cognitively connect parts of the solution. Concretely, during this process the students are led to cognitively connect additional, complementary, transformational reconstructions of the problem configuration and actions aimed at externalizing the student's thoughts by means of suitable chain questions which guide them towards the solution to the problem.

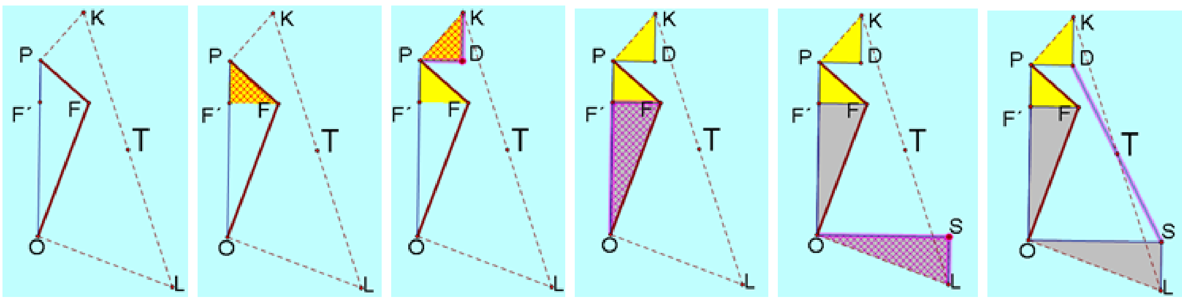


Figure 5: Example from the first problem - the sequential phases of the LVAR

Example from the first problem: The students progressively observe the rotations by 90 degrees of the similarly coloured triangles and the construction of segment DS (Figure 5). The students are led to shape an instrumented action scheme relating to the rotation of segments PF and FO.

The dialogue that follows is indicative of the student's construction of a section of the proof [55]:

214. Student M_2: *PDSO is a trapezium* ***because*** *PD and SO are perpendicular to PO, as we concluded from the rotation through* 90^o. *…**we must prove** that T is the midpoint of any segment … Should we join K and S? …. …**If we prove** that KL, DS are the diagonals of a parallelogram,* ***then*** *the diagonals are dichotomized….. …* (Figure 5)

216. Student M_1: ***if*** *we prove that these are parallel lines* ***then*** *the quadrilateral is a parallelogram* ***because*** *these are equal,* ***so*** *the diagonals will be intersected,* ***so*** *the diagonals will be dichotomized*

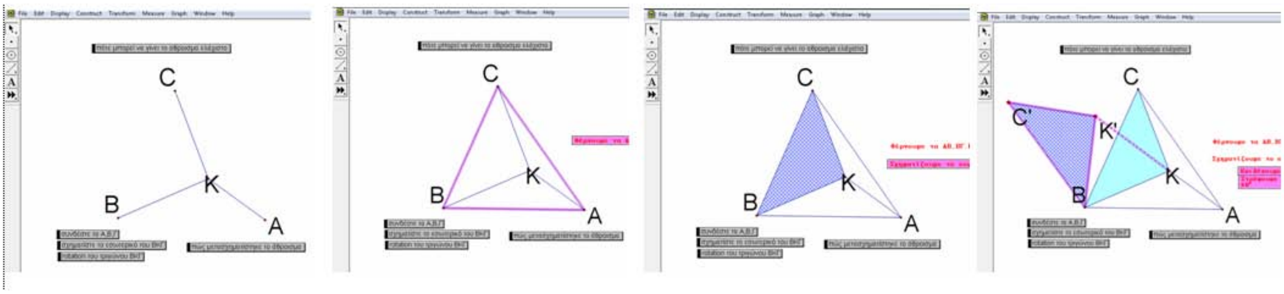


Figure 6: Example from the second problem - the sequential phases of the LVAR

Example from the second problem: The action buttons provide the student with a sequence of progressive instructions: "Connect points A, B, C", "Construct the interior of the triangle KBC", "rotate the triangle KBG" or "How has the sum been transformed?"

By pressing the buttons, the student can see the following executed simultaneously: A constructional process on the on-screen diagram and a computational process in which the sum of the segments is transformed. Use and manipulation of the action buttons makes it possible to link the following forms of representations—figurative/iconic, symbolic and verbal—which appear almost simultaneously on screen. The questions on the buttons point out that the process supplements rather than replacing the teacher, since the teacher initially prompts the students to explore/experiment and intervenes with a question essential for understanding the transformation. For example, in the question "how can we display the sum of the segments on a line as collinear points?" the students could be guided by pressing the first button which will display the rotation of the triangle through 60^o (Figure 6). "The empirical experiences are broadened though manipulations. These manipulations have been sufficiently mastered by the students and they are accompanied by a more conscious perception in a geometric sense" (Dina van Hiele in [22]).

During this process, a geometrical object is transformed into a new geometrical object emanating from the rotation (This process leads to the transformation of the sum of the three segments AK, KK΄ and K΄B΄ on a crooked line) and is followed by a mental transformation. That is to say, the process begins in the spatiographical and leads to the theoretical field. As Olivero writes, "A condition that can help the *focusing process* is the possibility of having a field of experience which allows students to manipulate, interact, and change the objects they deal with: such an empirical experience is likely to evoke theoretical elements" ([53], p.274). In this particular phase, the students become familiar with the basic links in the nexus of relations that take shape.

Throughout his/her teaching, the teacher organizes the activities for the special cases or actions that are expected from the students. The teacher can also simultaneously prepare the transformation in the iconic and symbolic representation, highlighting the different steps/strands in the solution in different colours, rendering the reaction evoked from the on-screen stimulus.

6.3 Mode C – the explicitation mode

Transformations in increasingly complex linked dynamic representations of the same phase of the problem under investigation modify the on-screen configurations simultaneously when, for example, a point is moved or has its orientation changed using the dragging or other tool. According to Dina van Hiele (see [21]) "The material has to be representative in the sense that it allows the context to become clear. A figure undergoes a *metamorphosis* as a result of the manipulations followed by a phenomenological analysis and an explicating of its properties: it becomes what we call a geometric symbol" (Dina van Hiele in Fuys et al. in [22]). The students explore the problem and start using *dragging*, meaning *a basic interaction*. They can observe a *continuous flow* on the screen because "cause and effect are observed simultaneously" ([64] p.7).

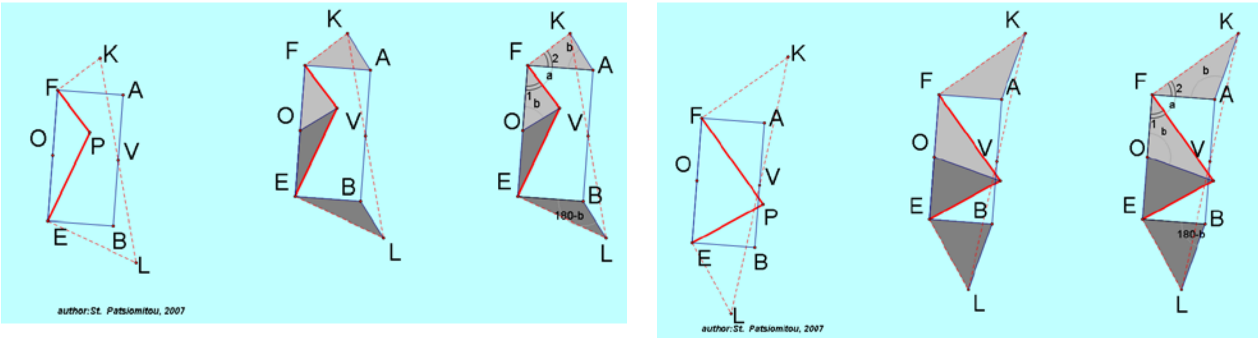


Figure 7: The transformed phases of the LVAR (problem 1)

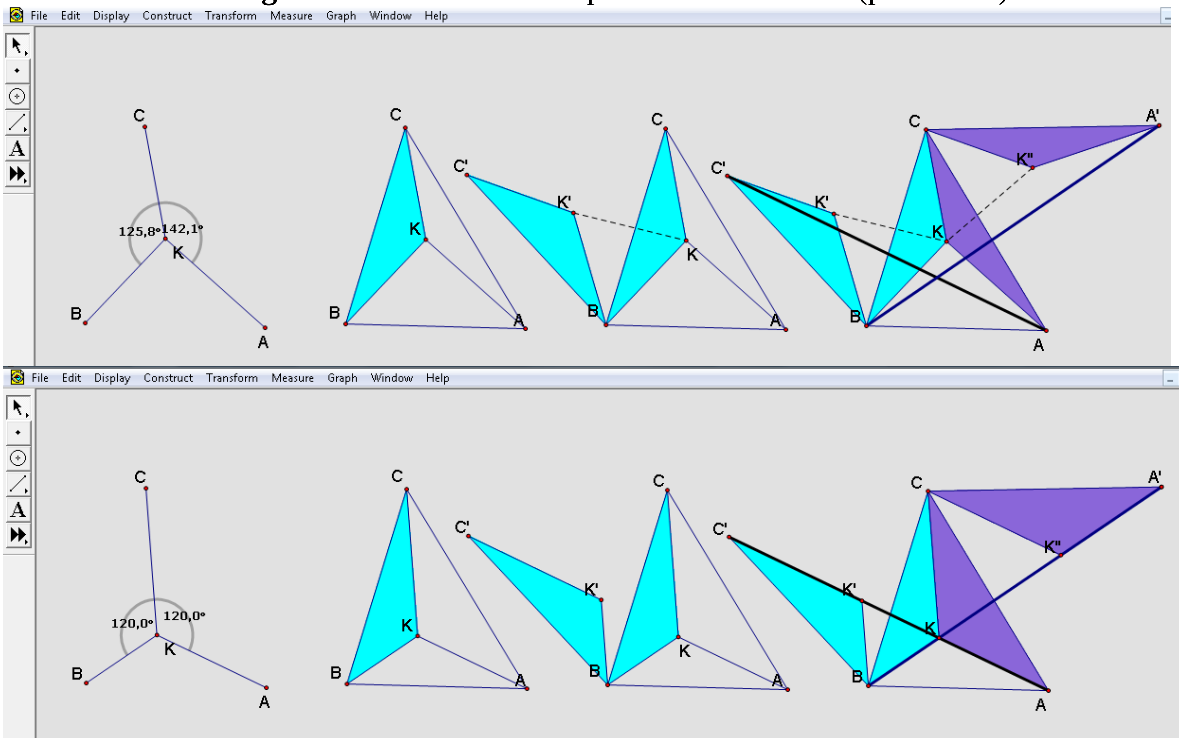


Figure 8: The transformed phases of the LVAR (problem 2)

Examples from the first and second problems: The successive phases of the constructional steps have been achieved using transformational processes like the use of the translation command (Figures 7 and 8). By dragging a point of the original configuration or the translated images, the students can observe the processes that emerged previously being modified simultaneously. Students are able to directly assume or infer the properties and the interrelationships between figures from properties indicated on the diagram by *conventional marks* (for example the equality of angles, or the angles measurements). The process leads the student to construct an infinite class of transformational processes of the same geometrical object on screen, and consequently to a generalization of the conclusions they have been led in previous phases of the solutionAccording to Dina van Hiele “The results of the manipulation of material objects are now expressed in words. The figures acquire geometric properties—so the goal of explication is to establish properties of figures. As a result, the shape becomes less important and the figure become a conglomerate of properties” [22]**.**

6.4 Mode D –the free orientation mode

Every phase in the solution can be displayed side by side on the same page of a sketch by pressing the action button which presents the global configuration rather than complementary parts of the configuration.

The students can focus their observation on what extra information is presented in the next emerging iconic form of the representation. The emerging additional representations can be dragged independently; for example, dragging the vertices of the triangle in configuration 3 leaves configurations 1, 2, 4, 5 and 6 unmodified (Figures 9 and 10). “Showing construction lines, together with dragging the figure, will help the students to keep in mind the properties of the construction. Hiding some elements may be useful when wanting to focus on some particular configuration” ([53], p.279). The students explore the problem and start *using task-based interactions* like *filtering, rearranging, annotating, probing* parts of the figure, etc.

The students are led to a proof that confirms their initial reasoning, conjectures and exploratory processes. We could call this the intermediary phase between the guided phase and free orientation.

They are thus led to discover actions in the software in order to be led to the subsequent free orientation phase. The explanation phase is the phase in which procedural knowledge is transformed into conceptual knowledge—which is to say into proof; the phase in which process is transformed into meaning.

Example from the first problem: The first thing to appear in the shape consists of the outlines of the figure that results from rotating segments PF, FO through 90 degrees. Next, the congruent triangles are highlighted in the same colour, followed by the equal segments or equal angles. The action button under each configuration helps the students gain an overall grasp of the modifications to the shapes in the new configuration.

For instance, the student is led to produce the following discussion after all six images have been revealed to her (Figure 9):

232. M_7: Segments MK and PF′ are equal ***because*** *triangles MKP and F’PF are congruent because they* ***are right triangles*** *(mental scheme)… with KP = PF, and angle MKP* ***equal to*** *angle F’PF – because* ***angle*** *KPF′* ***is external*** *to triangle MKP so it is equal* ***to the sum of*** *angle MKP and a 90^o angle; but at the same time, it is constituted from an angle of 90^o and angle FPF′* (Figure 9).

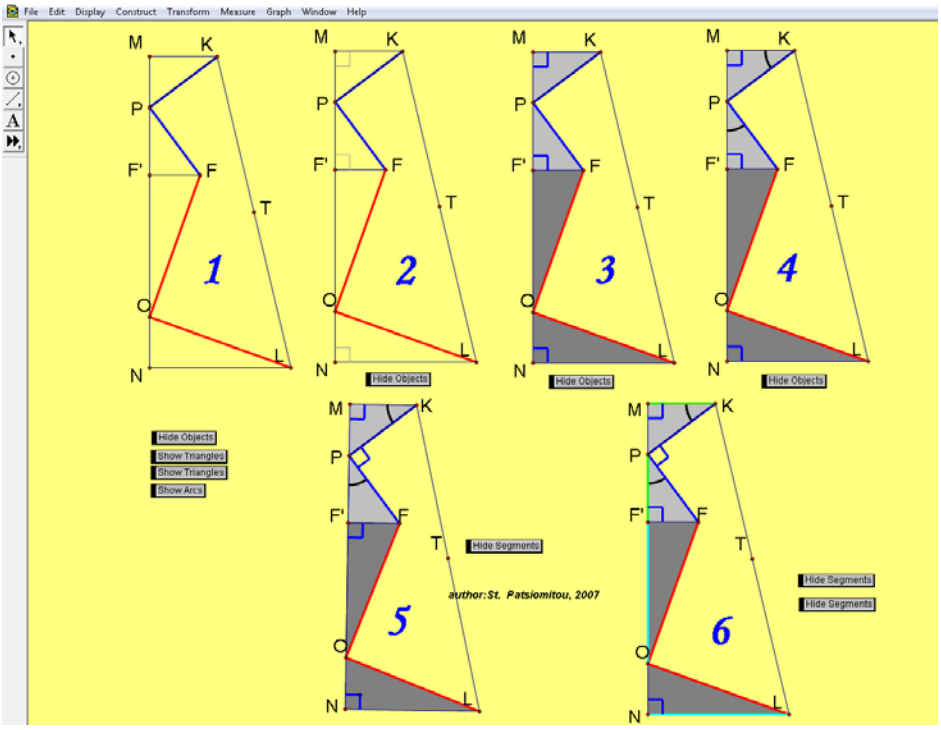


Figure 9: Example from the first problem

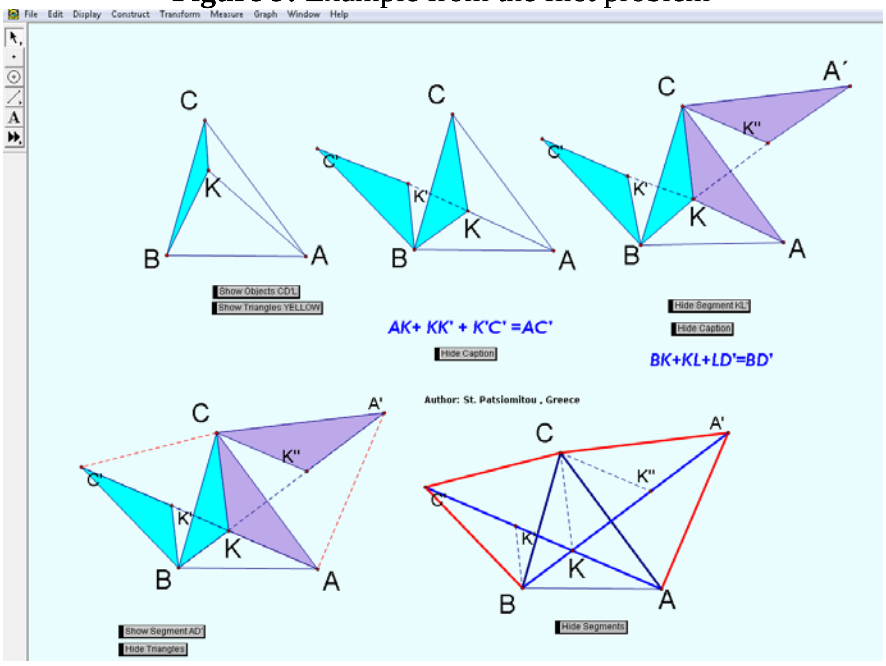


Figure 10: Example from the second problem

Example from the second problem: On the screenshot, we can see the emerging representations in the global diagram in which the student can recall key steps in the solution of the problem under investigation (Figure 10). It is essential that the student can display every step in the solution together on the same screen; only thus, can they see the progressive changes globally. A difficult problem is thus simplified through the use of pictures. The free orientation phase contains the translation of the proving process into condensed actions in the software.

"The field of investigation is for the most part known, but the student must still be able to find his way rapidly"(Dina van Hiele in [22]). The students can use their creativity to pose open goals with multiple steps and alternative solutions, thereby extending their knowledge to what they have seen before. One could call this the second phase of directed orientation, in which the students learn to find their way through the network of relations assisted by their extant knowledge. For example, the proving process leads to a solution which requires the construction of the circumscribed circles of the equilateral triangles with a view to finding their intersection, which is the solution to the modeled problem (Figure 13). This means continuous transformations between the theoretical and spatiographical fields.

6.5 Mode E –the integration mode

The solution to the problem [55] in global terms consists of successive configurations on different pages; configurations that are connected cognitively though not necessarily constructionally. This process is linked to the strategies for solving the problem or foreseeing the different strands in the solution relating to individual thought processes or different goals. This process can help students progress through the successive steps in the solution to completion.

During this phase, the student "must still acquire an overview of all the methods which are at his disposal. Thus he tries to condense into one whole the domain that his thought has explored. At this point the teacher can aid this work by furnishing global surveys. It is important that these surveys do not present anything new to the student; they must only be a summary of what the student already knows" **(**Dina van Hiele, [22]). This means that the information with which they became familiar in the new network of evoked geometrical objects and their interrelationships is reviewed and summarized. The students have developed thinking processes and applied skills, developing a mathematical model to interpret the realistic problem.

Example from the first problem (Figures 11, 12, 13):

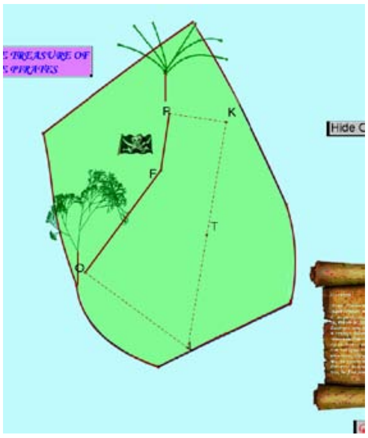


Figure 11: The problem in its initial modeled representation

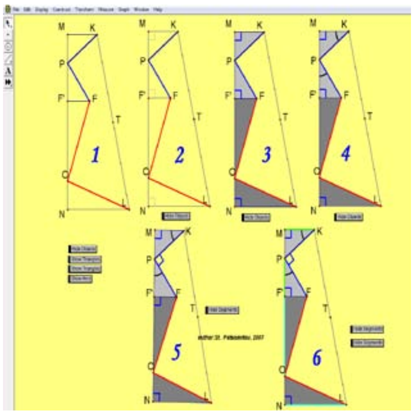


Figure 12: The intermediary phases of the problem, all in one representation

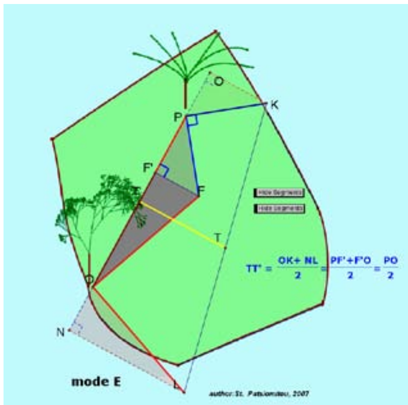


Figure 13: The solution to the problem in the final modeled representation

After the students have acquired an overview of Modes C and D, student M_8 in Mode E says:

Student M_8 : *the quadrilateral KONL is a trapezium and TT΄ is equal with the sum (KO + LN)/2. But KO + LN= PF΄+F΄O =PO,* ***so*** *the segment TT΄ is* ***the half*** *of the segment PO .*

Example from the second problem (Figures 14, 15, 16): The students are guided to an interpretation of the process in the modeled problem. At this stage, if the students are guided correctly, they must have examined every previous step successfully. For example, students can apply the custom-tool "construction of the circumscribed circle" to the sides of the equilateral triangles, so that the intersection point of the three circles gives the right place for point K, which is the solution and the interpretation of the solution to the real problem. With the definition and use of the "construction of the circumscribed circle" custom-tool, the students developed a *conversation* with the diagram on the screen using a *basic interaction*. They have progressed from a general rule and presented results relating to the particular, inferred case. After the students have acquired an overview of Modes C and D, student M_1 in Mode E constructs the triangles directly on the map. He says:

Student M_1: *There is no need to construct the circles, only the equilaterals. Then we have to join the opposite points.*

This transcript indicates that the student was verbalizing the cognitive connection between modes D and E.

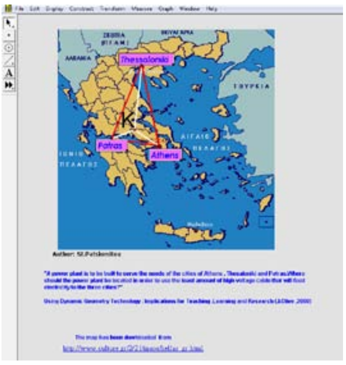


Figure 14: The problem in its initial modeled representation

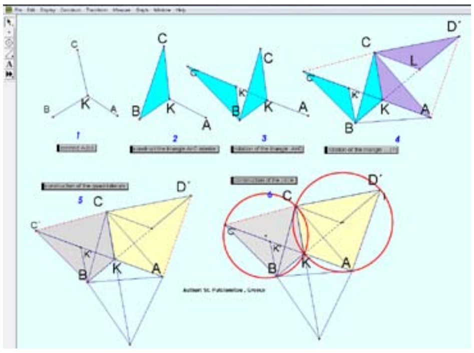


Figure 15: The intermediary phases of the problem, all in one representation

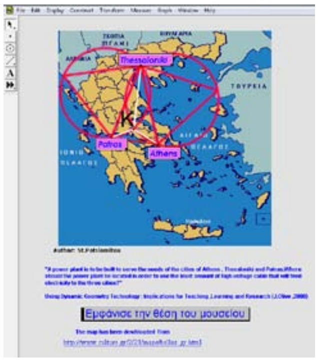


Figure 16: The solution to the problem in the final modeled representation

This process is a combination of advanced actions in the software and the proving process or strict justification; meaning that a software process has been transformed into a theoretical process by condensing the steps and using the custom tools facilitated by the Sketchpad software to prove that the point whose location they have to find is the point at which the circumscribed circles intersect.

7. Conclusion

According to Laborde [43], "Dynamic Geometry environments break down the traditional separation between action (as manipulation associated to observation and description) and deduction (as intellectual activity detached from specific objects) and reinforce the moves between the spatial and the theoretical domains." When the instrumental genesis occurs, transformations of linking representations globally or on the objects in the LVAR (i.e. artefacts or tools in the software) reflect on the assimilation or the accommodation of the situation by the subject. The students' development of geometrical thought takes place through the interaction with the LVAR in relation to the progressive adaptation of their schemes of use.

Therefore, it appears that the use of LVAR in the Sketchpad dynamic geometry environment proving process can organize the problem-solving situation using as tools the interaction techniques facilitated by the software, and the structuring and restructuring of the user's instrumental schemes it evokes as the activity unfolds. As the LVARs' composition changes, there is a transformation of the user's verbal formulations due to rules subjacent to the user's organized actions. Consequently, the scheme of use associated with the constructed instrument changes leads the students to pass from an empirical to a theoretical way of thinking or to students' mental transformations (Figure 17).

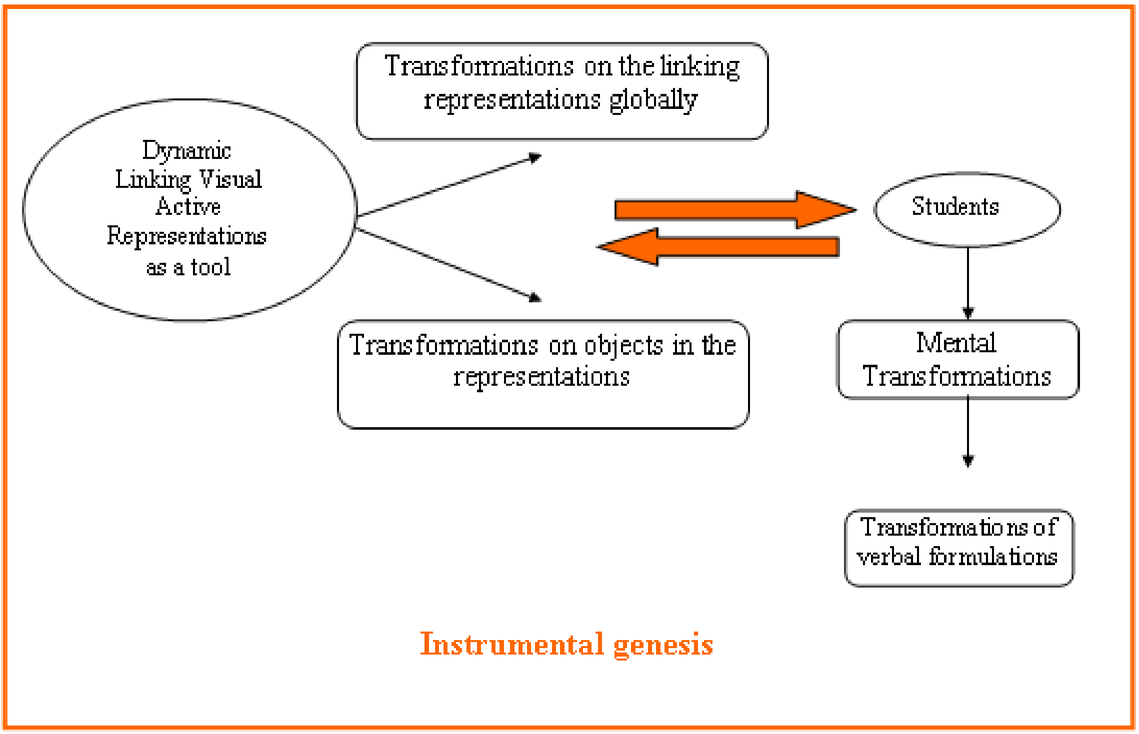


Figure 17: The transformations that occurred to students during their interaction with LVAR

Mathematical properties can be described in terms of transformations which may be represented through several types of manipulative activities. In the case of modelling a problem in the DGS environment, this process can be achieved through interaction techniques in the software during the problem-solving process. Initially, the students perform actions upon semi-predesigned LVAR. But eventually when the LVAR as objects become distinct images, students are able to perform mental transformations upon these images in a cognitive operation which builds upon actions but goes beyond them. During the interaction with LVAR, two different developments occur simultaneously: One is vision-spatial, using processes on the screen to perform tasks (i.e. rotation) that are completed between a pre-image (the original figure before transformation) and an image (the corresponding figure after the transformation). The other is conceptual, using concepts (i.e. properties of figures, interrelationships between figures, theorems etc.) and verbalized thoughts. In other words, the interaction with LVAR becomes a versatile connection between visual and mental objects [67]. The process of proof is developed using verbal formulations and geometrical relationships which become conceptualized during the proving process. Students use verbal formulations to exchange their ideas. They transform their mental objects into a language mapping, corresponding to motion transformations on the sketch. *Semperasmatically*, actions on LVAR (or interaction with LVAR) leading to proofs also lead to the development of geometrical thoughts. Students can develop their level of thinking by proceeding through increasingly complex, sophisticated and integrated figures and visualizations to a more complex linked representation of a problem, and thereby moving instantaneously between the successive Linking Visual Active Representations by means of their mental consideration and without returning to previous representations to reorganize their thoughts [55].

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