Alternative Solutions to the Problem Corner – October 2021 issue

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Summary: We provide alternative solutions by using *GeoGebra Discovery* (*https://github.com/kovzol/geogebra-discovery#readme*) for both Problems 1 and 2 (Problem Corner, Oct. 2021).

Problem 1. Let $X_1, X_2, ..., X_n$ be the vertices of a regular n-gon P and let P be any point interior to P . We denote by P_{ij} the projection of P onto $r(X_i, X_j)$**.** By abuse, we denote $X_{n+1} := X_1$, and so $P_{n,1} := P_{n,n+1}$. Prove that the sum

$$\sum_{i=1}^{n} X_i P_{i,i+1}$$

is constant, that is, it does not depend on the point P.

SOLUTION

Let us use GeoGebra Discovery see [Kovács-Recio, 2020], to address Problem 1. We start considering the case of an equilateral triangle (see Figures 1 and 2). In Figure 1 we ask GeoGebra to prove the equality between the sum of the three segments l, m, n, and the semi-perimeter 3/2 *f, where f is the side of the equilateral triangle. The literal answer is that the statement is true except when some of the following list of equalities happens:

l1 = {true, {"AreEqual[A,B]", "f * 3 + n * 2 = l * 2 + m * 2", "l * 2 + f * 3 + n * 2 = m * 2", "l * 2 + n * 2 = m * 2 + f * 3", "m * 2 + f * 3 + n * 2 = l * 2", "m * 2 + n * 2 = l * 2 + f * 3", "n * 2 = l * 2 + m * 2 + f * 3"}}

The first equality is A = B (a degenerate triangle). The remaining ones can be more clearly expressed as follows: {"(3f - 2(l + m - n)) = 0", "(3f - 2(- l + m - n)) = 0", "(3f - 2(l - m + n)) = 0 ", "(3f - 2(l - m - n)) = 0 ", "(3f - 2(- l + m + n) = 0", "(3f - 2(- l - m + n)) = 0"} and correspond to all possible sign choices for l, m, n except l + m + n and - l - m - n. Of course, the case l + m + n is precisely the one we are checking for its validity. And the case - l - m - n is not geometrically meaningful, since f is a segment length (thus positive) and - l - m - n is negative, for the same reason.

Thus, the six cases that GeoGebra indicates should be avoided for the truth of the given statement correspond to the choice of different signs for the variables describing the length of a segment, as they are defined algebraically by means of a degree two equation (e.g. if $C=(c_1, c_2)$ and $E=(e_1, e_2)$, $l^2 = (e_1-c_1)^2+(e_2-c_2)^2$) and there is no way for being more precise in the context of computational complex geometry, that is underlying GeoGebra Discovery implemented algorithms. Otherwise, we should have to work on the realm of computational real algebraic geometry, more precise but, currently, less performing. It must be noticed that this implies, in some sense, the need to make statements that include only even powers of the variables concerning lengths of segments. Thus, if one introduces —as done in this problem—a thesis such as $3f - 2(l + m - n) = 0$, it will be automatically converted to the product of all eight similar expressions, playing with the signs of l, m, n:

$(3f - 2(l + m - n))(3f - 2(- l + m - n))(3f - 2(l - m + n))(3f - 2(l - m - n))(3f - 2(- l + m + n)(3f - 2(- l - m + n))(3f - 2(l + m + n))(3f - 2(- l - m - n))) = 0$

involving the following degree eight homogeneous polynomial with only even powers in all the variables:

$$6561f^8 - 11664f^6l^2 - 11664f^6m^2 - 11664f^6n^2 + 7776f^4l^4 + 5184f^4l^2m^2 + 5184f^4l^2n^2 + 7776f^4m^4 + 5184f^4m^2n^2 + 7776f^4n^4 - 2304f^2l^6 + 2304f^2l^4m^2 + 2304f^2l^4n^2 + 2304f^2l^2m^4 - 23040f^2l^2m^2n^2 + 2304f^2l^2n^4 - 2304f^2m^6 + 2304f^2m^4n^2 + 2304f^2m^2n^4 - 2304f^2n^6 + 256l^8 - 1024l^6m^2 - 1024l^6n^2 + 1536l^4m^4 + 1024l^4m^2n^2 + 1536l^4n^4 - 1024l^2m^6 + 1024l^2m^4n^2 + 1024l^2m^2n^4 - 1024l^2n^6 + 256m^8 - 1024m^6n^2 + 1536m^4n^4 - 1024m^2n^6 + 256n^8$$

See [Kovács, Recio and Solyom-Gecse, 2019] for more details about this involved issue!

Figure 3 shows a different, simpler, approach with GeoGebra Discovery, in which the user just conjectures the truth of the statement. The reply is that it is numerically true and, by clicking on the *More*... button, one gets that the result is symbolically "true on parts, false on parts" as the truth or failure of the statement depends on the chosen selection of signs for the lengths, as remarked above. Obviously, in the most common case, as in this problem, for positive lengths, the statement is true! See also [Kovács, Recio and Vélez, 2019] for a complete description of this topic.

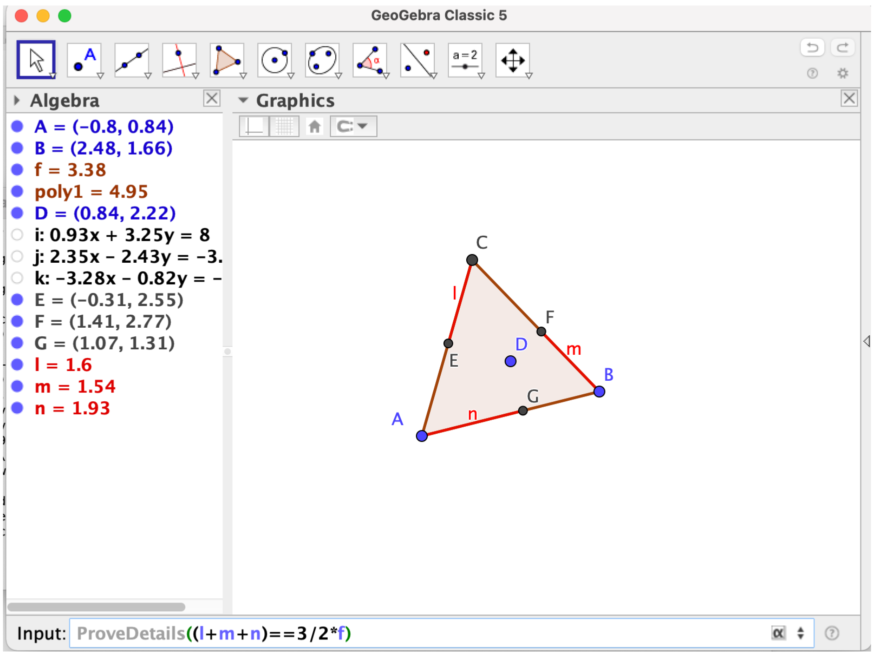

Figure 1: Asking GeoGebra Discovery to prove that the sum l + m + n is equal to the semi-perimeter 3/2* f.

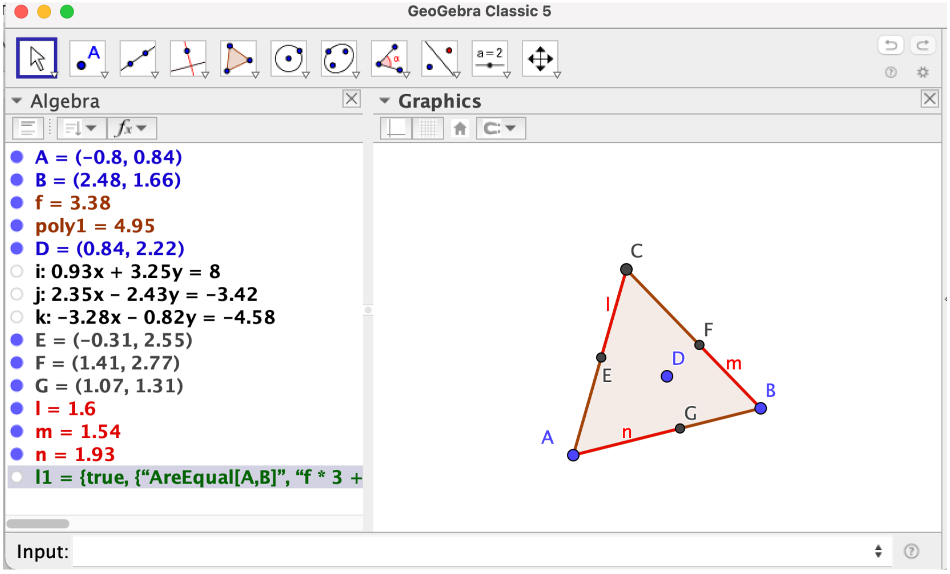

Figure 2: GeoGebra Discovery replies that the statement is true except in some degenerate cases.

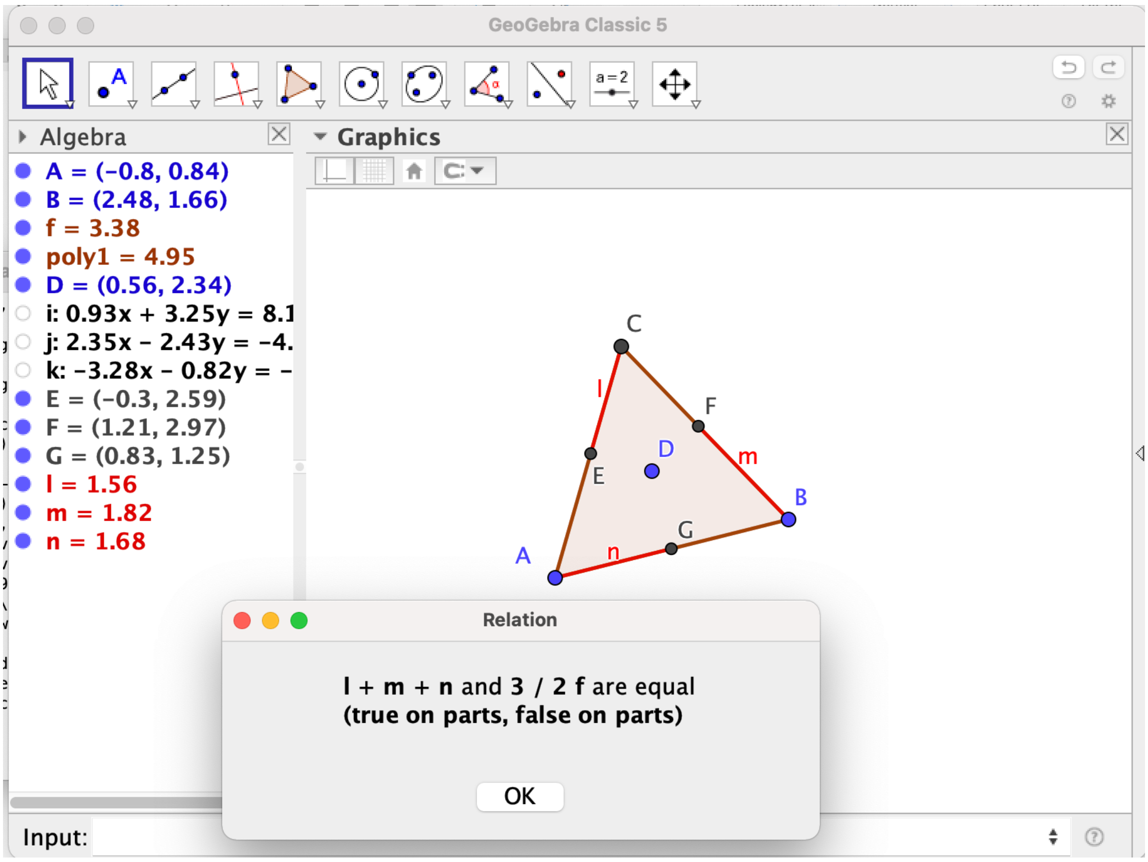

Figure 3: Verifying the truth of the statement with the *Relation* tool, answering that it is true just on some components of the hypothesis configuration (namely, excluding when the point D is placed out of the triangle).

The case of a square (see Figure 4) is again solved by GeoGebra via the *Relation* tool, but in this case the *ProveDetails* command is unable to present the whole list of avoidable cases.

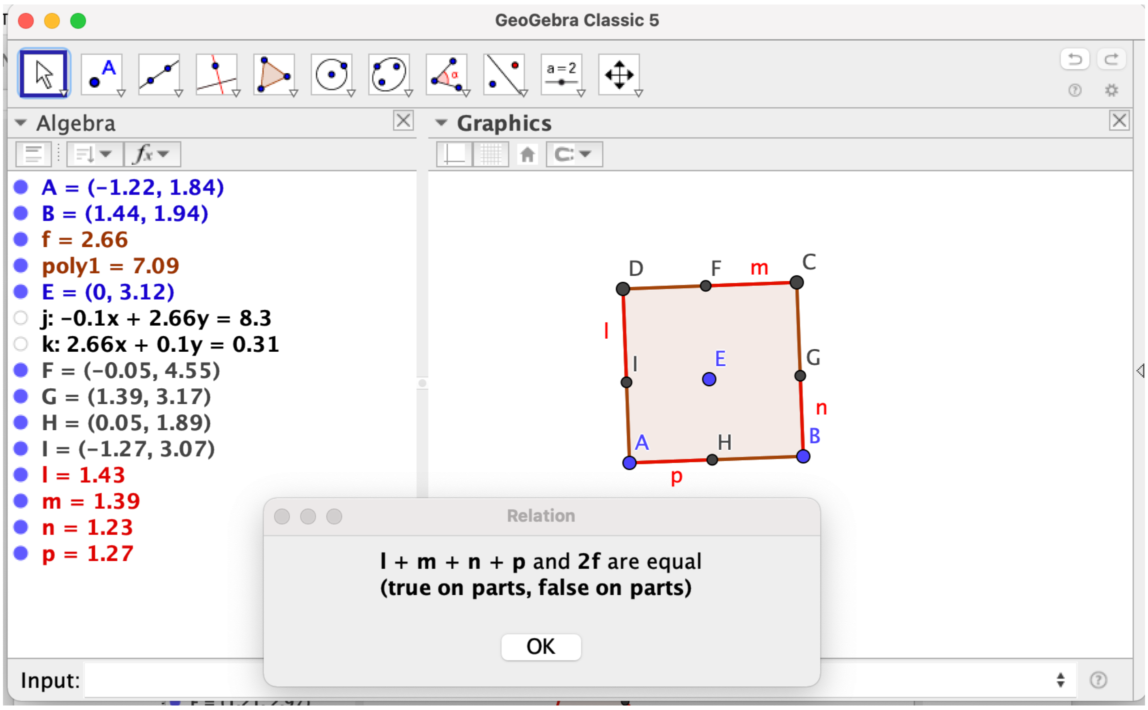

Figure 4. Verifying the truth of Problem 1 for the case of squares.

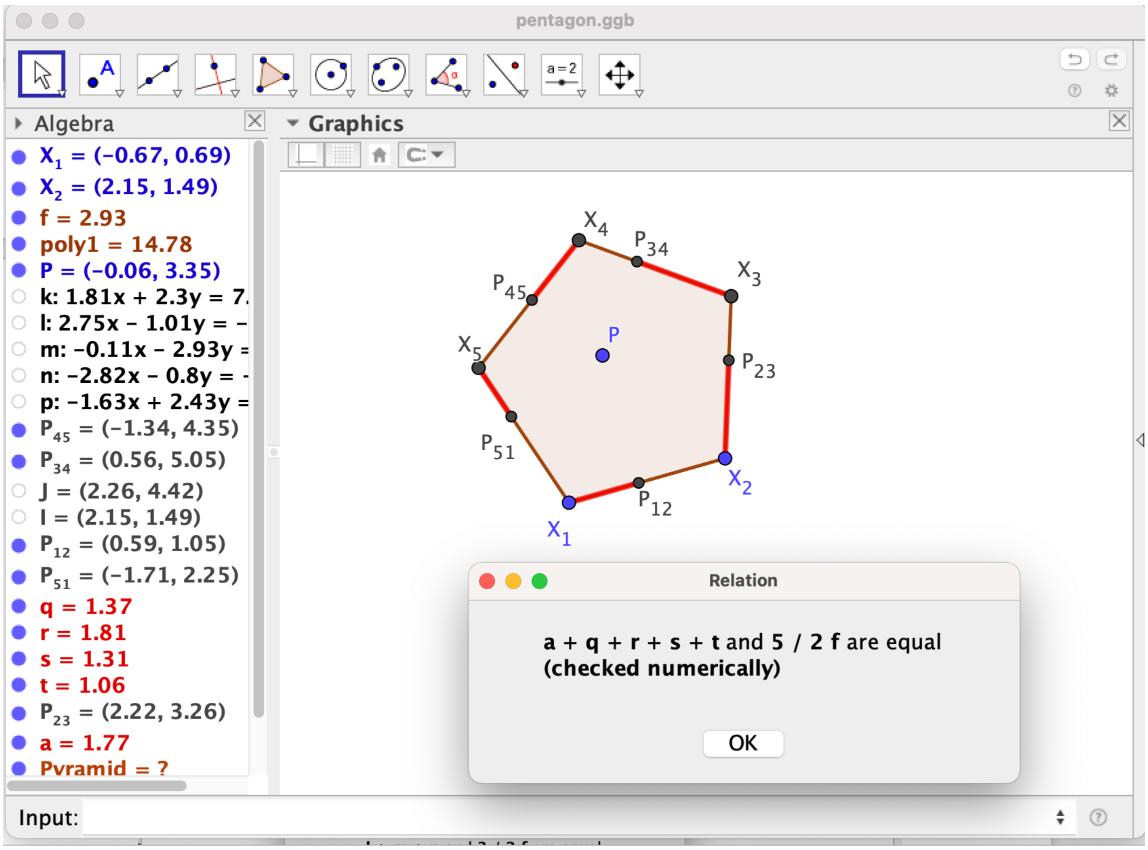

Figure 5. GeoGebra is unable to verify symbolically the truth of Problem 1 for the case of regular pentagons.

Finally, let us observe that the case of regular pentagons seems to be too demanding for the current version of GeoGebra Discovery, as the number of variables involved in the internal computer algebra algorithms increases too much, see Figure 5.

Problem 2

Let I be the incenter of a triangle ΔABC, that is, the point of intersection of the bisectors of the angles of the triangle. Let l_1, l_2 and l_3 be, respectively, the lines which are perpendicular through I to the lines $r(A,I)$, $r(B,I)$ and $r(C,I)$. Prove that the points

$$X := r(B,C) \cap l_1,\quad Y := r(A,C) \cap l_2 \text{ and } Z := r(A,B) \cap l_3.$$

are collinear.

SOLUTION

In this case the solution provided by GeoGebra Discovery is quite straightforward, see Figures 6 and 7, where it is shown that the answer to the *ProveDetails* or to the *Relation* command concerning the collinearity of the *X, Y, Z*, is that the statement is true except for the degenerate cases in which the triangle *ABC* collapses (ie. *A=B* or {*A, B, C*} are collinear).

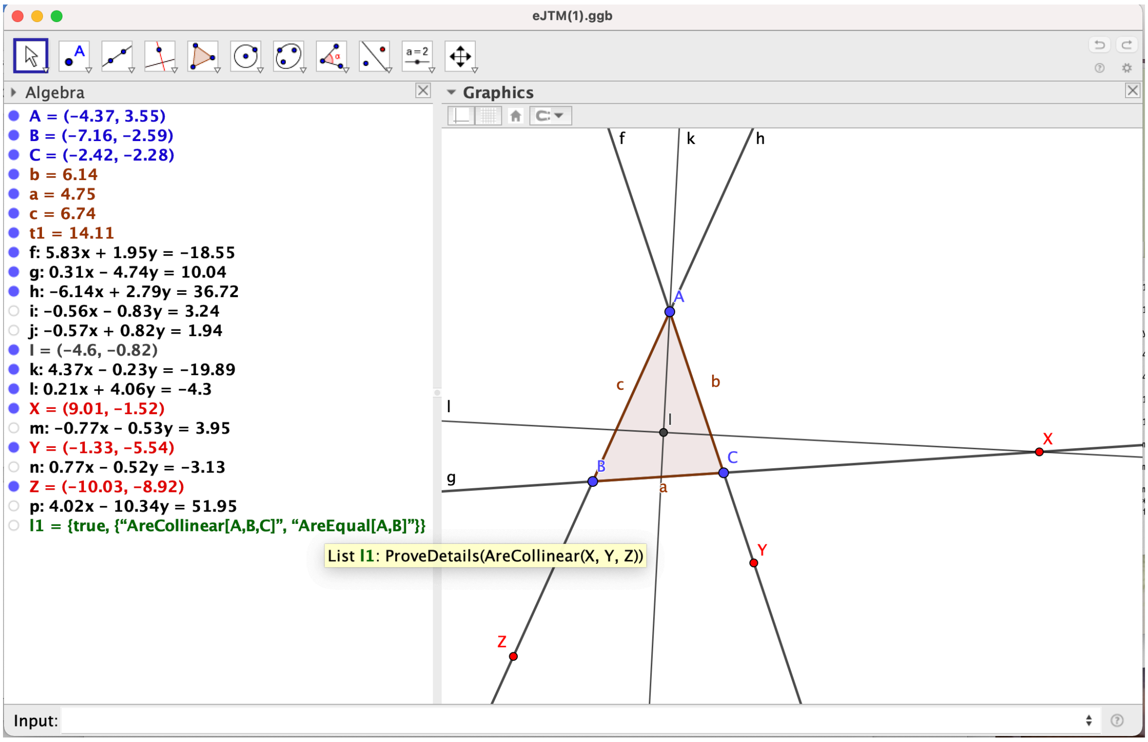

Figure 6: GeoGebra Discovery formal verification of Problem 2.

Notice that the main difference between both commands is that *ProveDetails* requires the user to input the precise statement, while *Relation* deals simply with two geometric objects (*Z* and *Line(X,Y),* in this case) introduced by the user and, initially expresses that, numerically, *Z* belongs to the line *XY* ; and as a second step, declares that *Z* lies (formally, not just visually or numerically) on this line, except when the triangle degenerates.

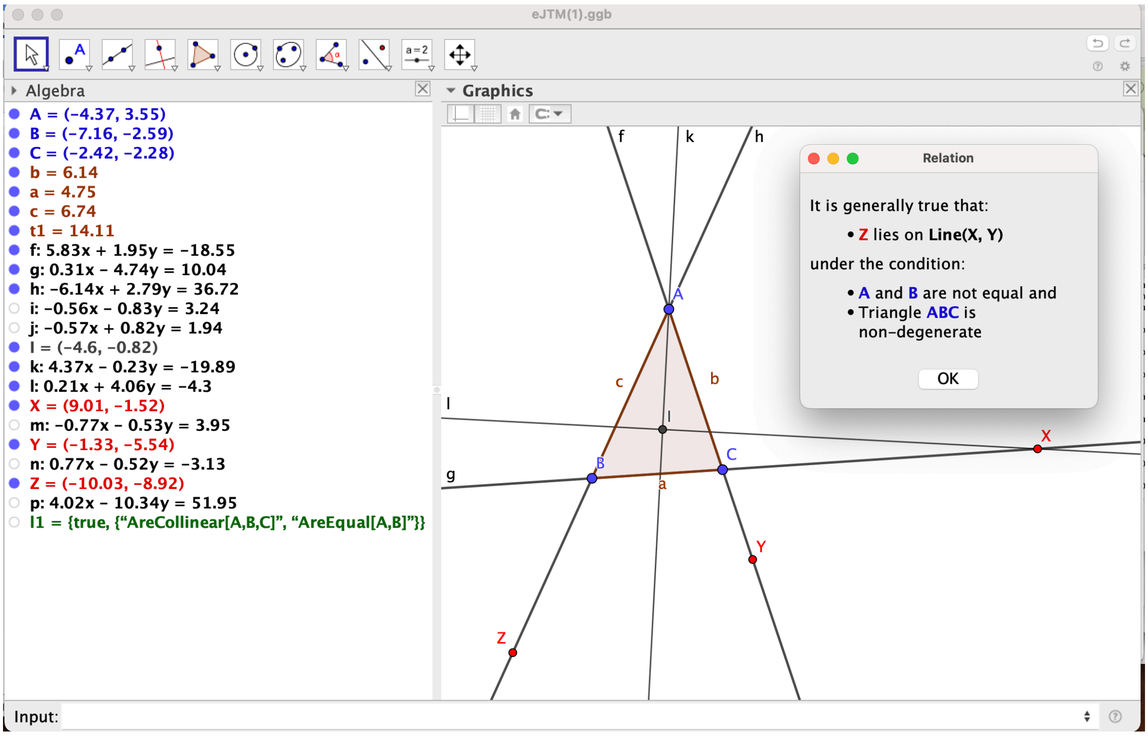

Figure 7: GeoGebra Discovery proof of Problem 2 through the *Relation* tool.

Finally, let us mention the quite recent *Discover* command in GeoGebra Discovery, that autonomously outputs a list of relevant properties involving a given geometric element. Figure 8 shows the output of *Discover(Z)*, that declares that *XYZ* are collinear and some other properties (such as the perpendicularity of *CI* and *IZ*, which is obvious in this case, as it is part of the definition of *Z*). Helping the *Discover* command to evaluate the interestingness of the obtained outputs is a quite involved, pending research issue.

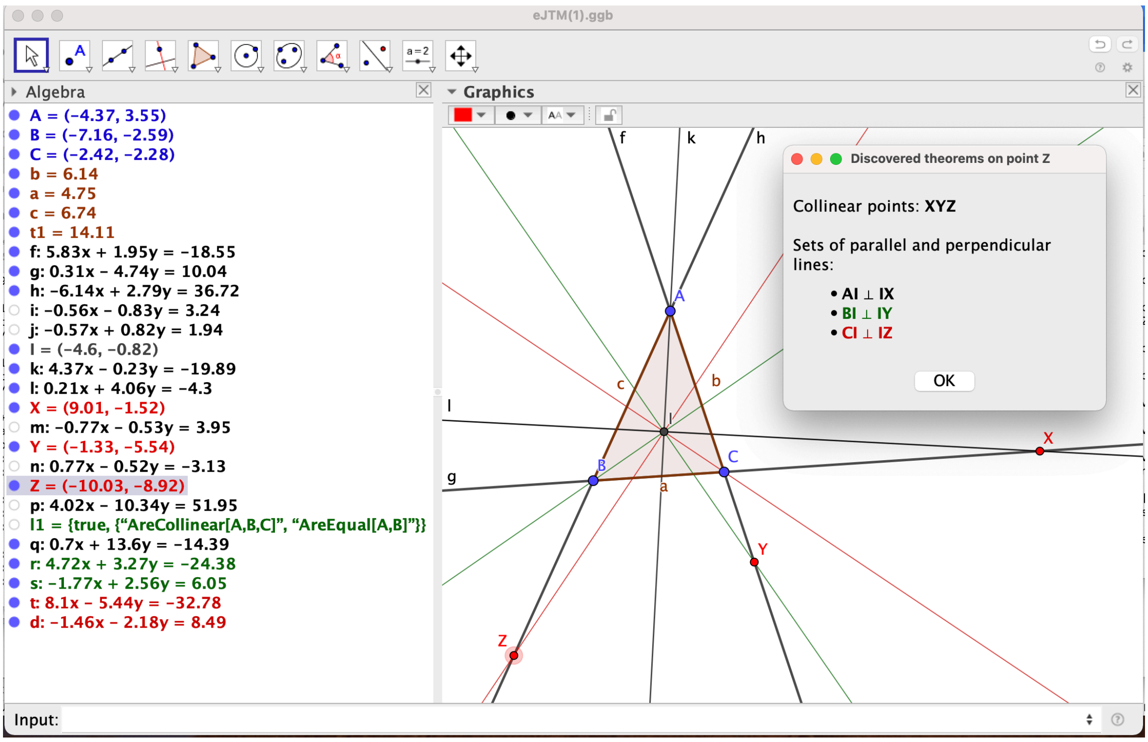

Figure 8: Automatically finding the statement in Problem 2 and verifying it, through the *Discover* command.

REFERENCES

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