PROBLEM CORNER

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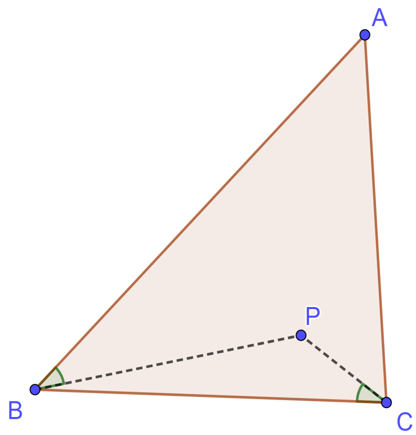
Introduction

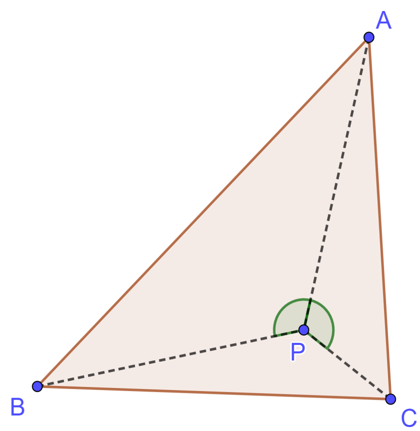
We solve two loci problems proposed by T. Recio and C. Ueno in the February 2025 Problem Corner issue. We use *Mathematica* [1] and the package *Baricentricas* written by the author.

Problem 1. *Consider a triangle ABC. Find the geometric locus of points P such that $\angle PBA = \angle PCB$ and study its properties.*

Problem 2. *Consider a triangle ABC. Find the geometric locus of points P such that $\angle APB = \angle CPA$ and study its properties.*



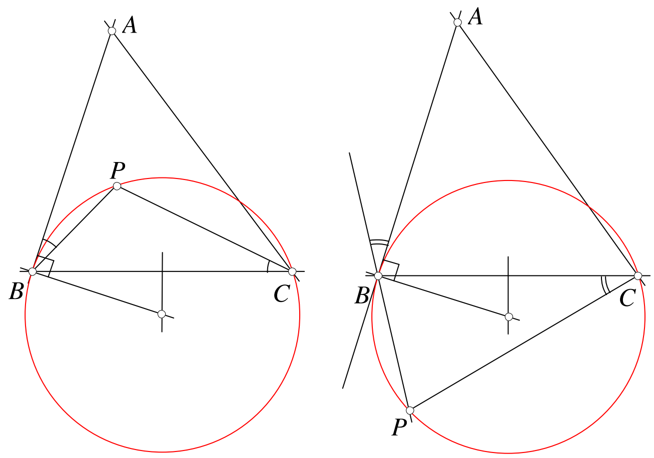
Problem 1



Problem 2

[1] https://www.wolfram.com/mathematica

Problem 1



(a) (b)

Solution by elementary geometry. Let $\mathcal{C}$ be the circle through B and C that is tangent to AB at B. If P lies on the circle and P lies in the same side of BC as A, then $\angle PBA$ and $\angle PCB$ are, respectively, semi-inscribed and inscribed angles that subtend the same arc of the circle, and thus they have the same measure.

This property remains valid when P lies on the side opposite to A, if we consider oriented angles, that is, if we consider $\angle PBA$ as the angle that the line BP must rotate counterclockwise around B to coincide with line BA.

As an alternative approach to solve this type of problems by algebraic computations, let us introduce, in the next section, some basic ideas about the rotation of lines in barycentric coordinates.

Rotations of lines: general formula

Let us consider here the general problem of rotation of lines. This problem was proposed by the author to Paul Yiu, chief-editor of *Forum Geometricorum*[2], an excellent journal on Classical Euclidean Geometry that was active during the two first decades of this century. We refer to [1] for notation and further details.

[2] https://en.wikipedia.org/wiki/Forum_Geometricorum

Given three lines in barycentric coordinate $\mathcal{L}_i : p_i x + q_i y + r_i z = 0, i = 1, 2, 3$, find a fourth line $\mathcal{L}_4$ such that $\measuredangle(\mathcal{L}_3, \mathcal{L}_4) = \measuredangle(\mathcal{L}_1, \mathcal{L}_2)$.

For any infinite point (u, v, w), with $u + v + w = 0$, we consider the infinite point

$$(u', v', w') = (S_B v - S_C w, S_C w - S_A u, S_A u - S_B v),$$

that satisfies:

1. (u, v, w) and (u', v', w') have perpendicular directions.
2. $S_A u'^2 + S_B v'^2 + S_C w'^2 = S^2(S_A u^2 + S_B v^2 + S_C w^2)$.

Now, if lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ have infinite points $(u_1 : v_1 : w_1)$, $(u_2 : v_2 : w_2)$, $(u_3 : v_3 : w_3)$ we can consider the infinite point $(u_4 : v_4 : w_4)$ defined by

$$\begin{aligned}(u_4, v_4, w_4) = S^2(S_A u_1 u_2 + S_B v_1 v_2 + S_C w_1 w_2)(u_3, v_3, w_3) \\ + (S_A u_1' u_2 + S_B v_1' v_2 + S_C w_1' w_2)(u_3', v_3', w_3')\end{aligned} \tag{1}$$

to get the infinite point of a line describing with (u_3, v_3, w_3) the same oriented angle as the one defined by (u_1, v_1, w_1) and (u_2, v_2, w_2).

This formula is used by the function `CuartaRecta` in the *Mathematica* package *Baricentricas* written by the author:

`CuartaRecta[ptP,r1,r2,r3]` returns the line through P forming with line r_3 the same angle as the one defined by the lines r_1 and r_2.

Problem 1 revisited

We can now use `CuartaRecta` to solve Problem 1, as follows:

```
ptP = {x, y, z};
Factor[CuartaRecta[ptB, Recta[ptC, ptP],
   Recta[ptC, ptB], Recta[ptB, ptP]]]
```

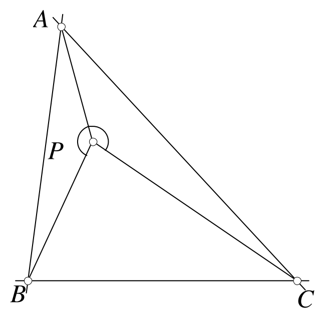
We get the equation $a^2yz + b^2xz + c^2xy - c^2x(x + y + z) = 0$ (where a, b, c are the lengths of the opposite sides to the vertices A, B, C) corresponding to a circle through B and C. Since the power of A with respect to the circle is c^2, we conclude the circle is tangent to AB at B, as previously observed.

Problem 2

Now let us address Problem 2. Note that the *Mathematica* function `Cross` that calculates the cross product of two vectors, returns $\{0, 0, 0\}$ if the two vectors are the same.

In this case the instruction that gives the locus is

```
Apply[PolynomialGCD,
 Factor[Cross[
   CuartaRecta[ptP, Recta[ptP, ptB], Recta[ptP, ptA],
    Recta[ptP, ptA]], Recta[ptP, ptC]]]]
```



Its output is the cubic

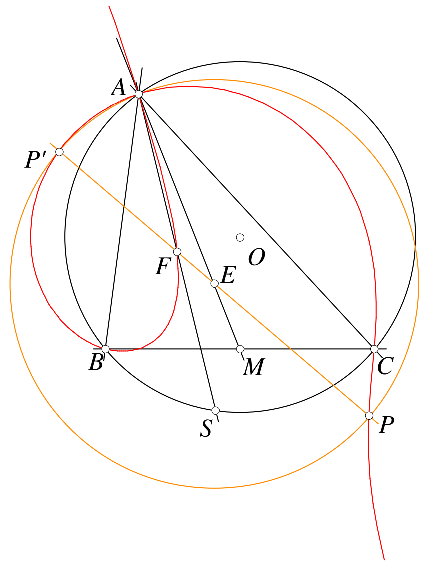
$$c^2xy^2 + a^2y^2z - b^2y^2z + c^2y^2z - b^2xz^2 - a^2yz^2 - b^2yz^2 + c^2yz^2 = 0,$$

a right strophoid through C with node at A. An inversion with center A and radius b gives the hyperbola

$$b^2xy + b^2y^2 - c^2y^2 - b^2xz = 0$$

through A and C, with asymptotes parallel to the bisectors of angle A and centered at $Q = (-b^2 + c^2 : b^2 : c^2)$, that can be easily constructed as the intersection of the $A-$symmedian and the $C-$sideline of the medial triangle.

Going back to the strophoid, it has an asymptote parallel to the median through A. If we take an arbitrary point E on the $A-$median, the circle through A centered at E meets the curve at two points P, P', and the line PP' goes through a fixed point F, the focus of the strophoid. The focus F has coordinates $(-a^2 + b^2 + c^2 : b^2 : c^2)$ and it is the midpoint of the segment AS, where S is the second intersection of the $A-$symmedian and the circumcircle. These points are displayed in the following figure, and computed through the described below instructions:




```
locus2 = Apply[PolynomialGCD, Factor[Cross[
     CuartaRecta[ptP,
      Recta[ptP, ptB], Recta[ptP, ptA],
      Recta[ptP, ptA]], Recta[ptP, ptC]]]];

ptE = {1, t, t};
circun = Numerator[Factor[
    Circunferencia[ptE,
     CuadradoDistancia[ptE, ptA]]]];

{ptP1, ptP2} = Map[Simplificar, {x, y, z}
    /. Drop[Simplify[Solve[
       {circun==0,locus2==0},{y,z}],x>0],3]];

ptF = -Simplificar[{x, y, z} /. Solve[{
       Recta[ptP1, ptP2].{x, y, z} == 0,
       locus2 == 0}, {y, z}][[1]]]
```

References

[1] F. J. García Capitán (2015). *Barycentric Coordinates*. International Journal of Computer Discovered Mathematics (IJCDM). November 2015, Volume 0, No. 0, pp. 32-48. https://www.journal-1.eu/2015/01/Francisco-Javier-Barycentric-Coordinates-pp.32-48.pdf