PROBLEM CORNER

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MOTIVATION

It is well known that Dynamic Geometry software is an excellent tool for teaching and learning geometry. See, for example, [4] and other chapters from the same volume where it is included. But in this Problem Corner we will not deal with problems to be solved by humans using Dynamic Geometry, but with *problems to be solved by GeoGebra automated reasoning tools*, available either at the standard GeoGebra version (www.geogebra.org) or in an experimental fork: *GeoGebra Discovery*, available in two options: GeoGebra Classic 5 (the one used in the following, *GeoGebra Discovery* version 2022May04, based on GeoGebra Classic 5.0.641.0-d), for Windows, Mac and Linux systems, that can be downloaded from https://github.com/kovzol/geogebra-discovery; and GeoGebra Classic 6, made for starting it in a browser at http://autgeo.online/geogebra-discovery/, mainly ready for use on tablets and smartphones. Details about the different available automated reasoning tools (*Relation*, *Prove*, *Discover*, *Compare*, *LocusEquation* commands, etc.) can be found at [5],[6]).

Thus, *the challenge here is for humans to help GeoGebra to solve the proposed problems* [7]. The context of both is the following fact: GeoGebra deals mostly with geometric statements that can be translated to algebraic equations (i.e., *not* involving inequalities). Although it is already possible to handle some inequalities (see [2]) it is on-going work to fully extend GeoGebra proving tools in this direction, given the high complexity (required amounts of memory and time) of such generalization. Thus, currently, we must think of some alternatives to approach, through GeoGebra, the proof of statements that include, for example, the bisector of an angle defined by two lines, as it is not possible to distinguish, without using inequalities, between the two possible bisectors associated to the two lines. This is the underlying issue concerning next Problem 1. A similar, even more involved, situation comes concerning Problem 2, where GeoGebra is faced with an optimization problem, where inequalities are implicit.

PROBLEM 1

Let I, O, H, denote the incenter, circumcenter and orthocenter of triangle ΔABC, respectively. Find necessary and sufficient conditions for the alignment of the three points.

We make the basic construction with GeoGebra (see Figure 1). Thus, f is the perpendicular bisector of AB, C is a point on f (so the triangle ΔABC is isosceles and side a= side b). Now consider g, the perpendicular bisector of side a, and let O be the circumcenter, i.e.

the intersection of f and g. Then build the line k through B perpendicular to AC, and let H be the orthocenter, as the intersection of f and k. Finally, consider lines l (bisector of angle(CBA)) and m (bisector of angle(BAC), and their intersection at the incenter I.

If we ask GeoGebra to find the relation between I and the Line(O, H), ie. Euler's line, GeoGebra is unable to give a rigorous, affirmative assertion, only a numerically approximate answer.

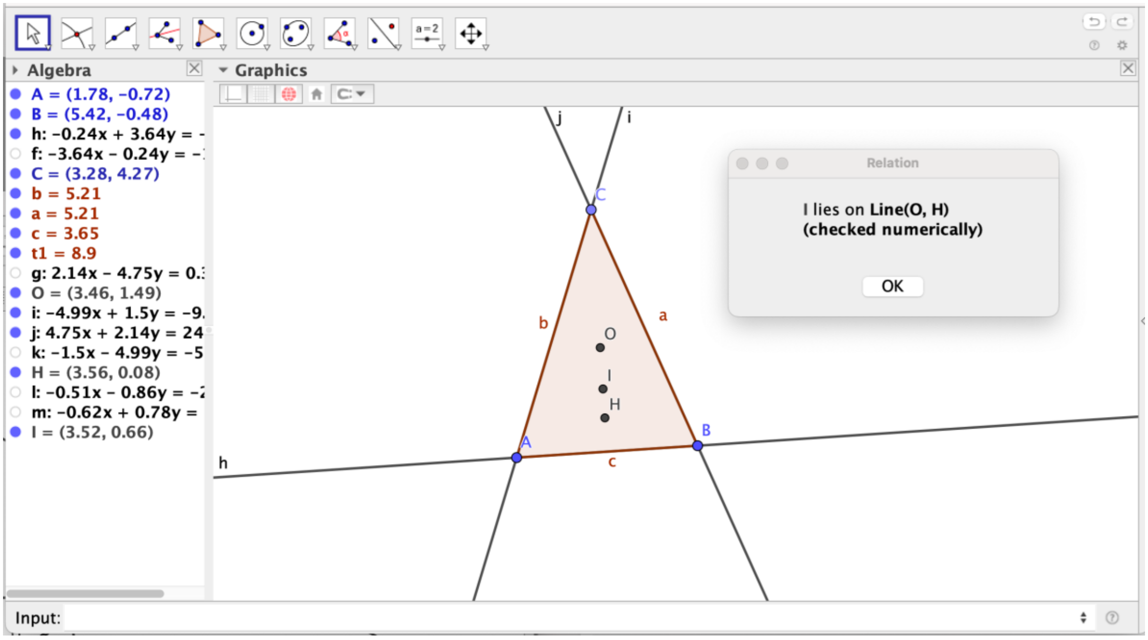

Figure 1: Given an isosceles triangle (a=b), O=circumcenter, H=orthocenter, I=incenter, *Relation*(I, Line (O,H)) does only answer approximately that I belongs to the Euler line.

It must be remarked that there is the option (in *GeoGebra Discovery*) to build directly the incenter I using the *IncircleCenter* command (that defines the incenter as the center of the circle tangent to the three sides of the triangle), but the answer to the *Relation*(I, Line (O,H)) is also only numerical in this case, as *IncircleCenter* is still under development and there are several circles tangent to the sides of the triangle: restricting the definition to the incircle that lies inside the triangle requires, again, to deal with inequalities. See [8] and [3] for a similar approach and related difficulties.

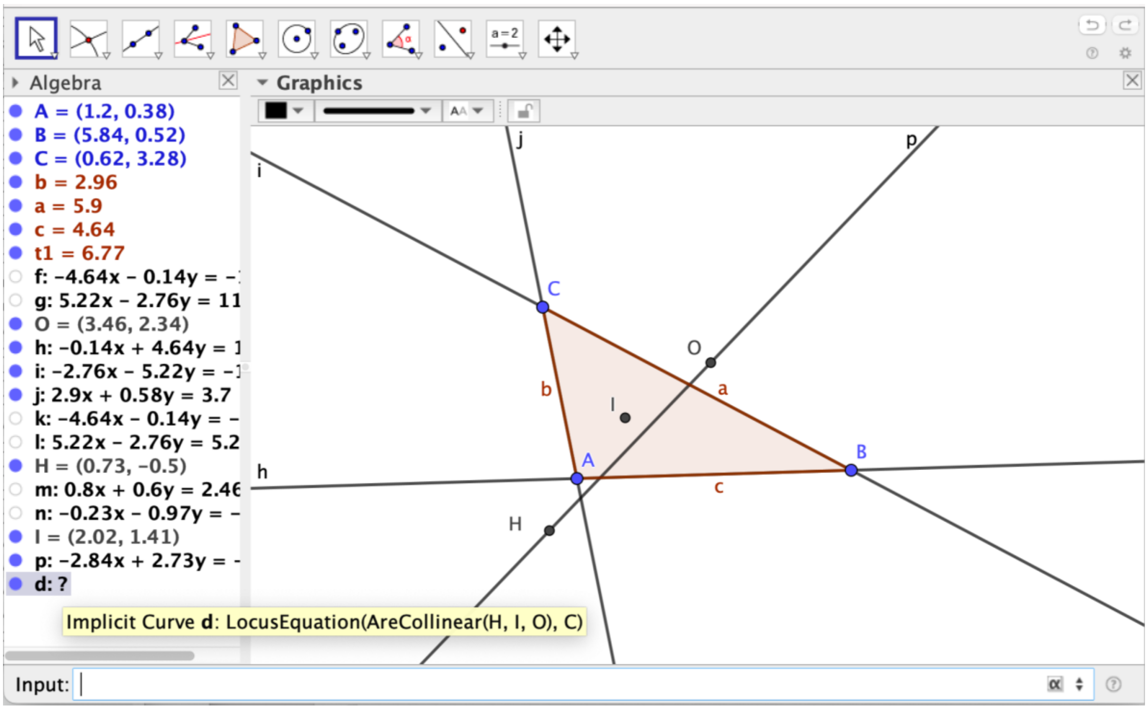

Figure 2: Conversely: LocusEquation of C for AreCollinear (H, I, O) yields "?".

Conversely, now we start with an arbitrary triangle ΔABC and we build, as the intersection of angle bisector lines, the Incenter I, the Orthocenter H and the Circumcenter O. Let p be the Euler line OH. Then we want to prove (see Figure 2) that if I lies on the Line (O, H), then the triangle must be isosceles, but GeoGebra ignores the locus of C for the triangle ΔABC to verify the collinearity of I, O, H.

PROBLEM: find an alternate way to deal with incenters that do not rely on signs, so that GeoGebra is able to find necessary and sufficient conditions for the alignment of the I, O, H.

HINT: Instead of starting with a triangle and then constructing the incenter, start with a given incenter and build the triangles having such incenter. Try, for the *LocusEquation* issue (Figure 2), to see if the *IncircleCenter* command helps at all.

PROBLEM 2

Prove that the equilateral triangles have the smallest perimeter possible containing a given area.

This is a classical problem, quite easy to solve with the traditional means, but not so simple to be immediately solved with GeoGebra. Indeed, there is no way to display with GeoGebra the set of all triangles with a given (symbolic) area, unless one fixes at least one side (say, AB, as in Figure 3). Then, minimizing the perimeter is not a task that GeoGebra can accomplish automatically; it requires –for the moment– some "human intelligence".

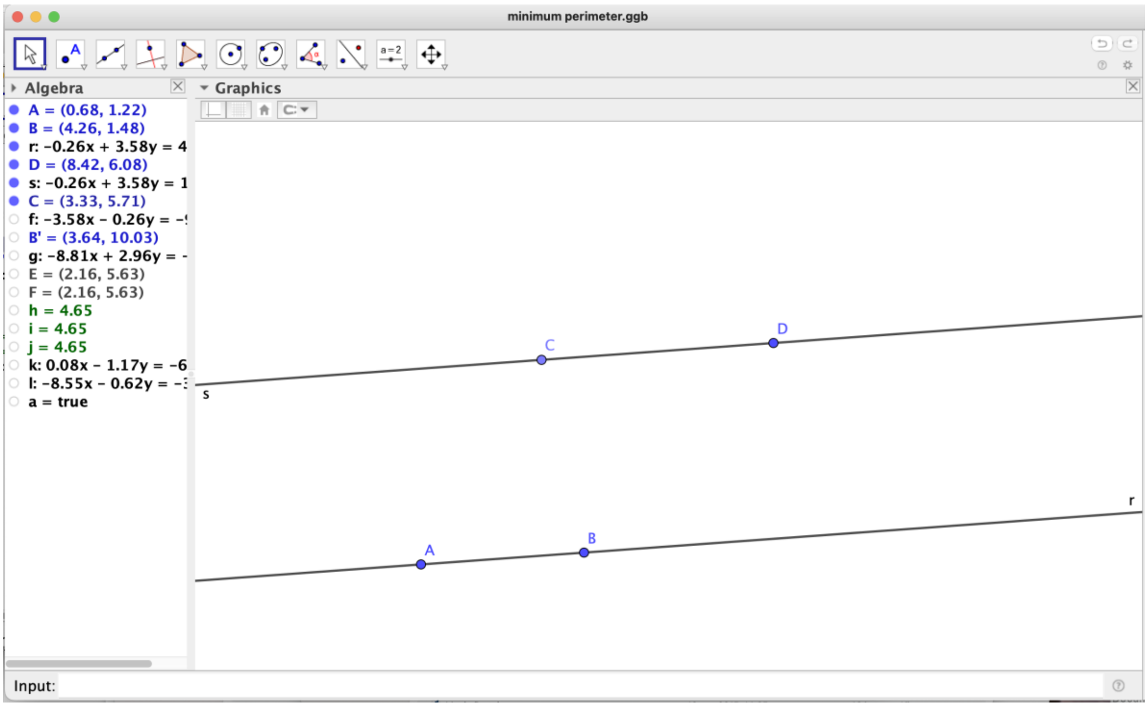

Figure 3: Side AB determining line r. Free point D and parallel line s to r through D. Point C in s.

PROBLEM: Given side AB and a parallel line to AB at a given distance, find (using GeoGebra) where to place vertex C in this line (so that the area of ΔABC will be equal for all positions of C) in such a way that the perimeter of ΔABC is minimum. See Figure 3.

HINT: Guess (as human) where to place C and have GeoGebra discovering that this position is the sought one.

Let us finish recalling the human readers of the eJMT 2022's, Kaput's visionary words [1]: technology should help us towards "a continuing transition from Doing (old) Things Better to Doing Better Things".

REFERENCES

[1] Balacheff, N.; Kaput, J.; Recio, T. ICME 8, TG19 Follow-up Report. Computer-Based Interactive Learning Environments: "CBILE".

https://web.archive.org/web/20150909222401/http://mathforum.org/mathed/seville/followup.html (accessed 05 April 2022).

[2] Brown, C.W.; Kovács, Z.; Recio, T.; Vajda, R.; Vélez, M.P. (2022): Is computer algebra ready for conjecturing and proving geometric inequalities in the classroom? In: Kotsireas, I., Simos, D., and Uncu, A.K. (Eds.), *Special Issue: Applications of Computer Algebra (ACA) 2021*. Mathematics in Computer Science, 2022.

[3] Dalzotto, G.; Recio, T. (2009): On protocols for the automated discovery of theorems in elementary geometry. *Journal of Automated Reasoning* 43: 203--236.

[4] Hohenwarter, M.; Kovács, Z.; Recio, T. (2019): Using GeoGebra Automated Reasoning Tools to explore geometric statements and conjectures. In: Hanna, G., de Villiers, M., Reid, D. (Eds.), *Proof Technology in Mathematics Research and Teaching*, Series: Mathematics Education in the Digital Era, Vol. 14, 2019, pp. 215-236. Springer Cham. https://doi.org/10.1007/978-3-030-28483-1_10

[5] Kovács, Z.; Recio, T.; Vélez, M.P. (2021): Approaching Cesàro's inequality through GeoGebra Discovery. *Proceedings of the 26th Asian Technology Conference in Mathematics*, W.C. Yang, D.B. Meade, M. Majewski (eds). Published by Mathematics and Technology, LLC. ISSN 1940-4204. Dec. 13-15, 2021. pp. 160-174. http://atcm.mathandtech.org/EP2021

[6] Kovács, Z.; Recio, T.; Vélez, M. P. (2022): Automated reasoning tools with GeoGebra: What are they? What are they good for? In: P. R. Richard, M. P. Vélez, S. van Vaerenbergh (eds): *Mathematics Education in the Age of Artificial Intelligence: How Artificial Intelligence can serve mathematical human learning*. Series: Mathematics Education in the Digital Era, Vol. 17, 2022, pp. 23-44. Springer Cham. https://doi.org/10.1007/978-3-030-86909-0_2

[7] Recio, T. (2022): Epilogue. In: P. R. Richard, M. P. Vélez, S. van Vaerenbergh (eds): *Mathematics Education in the Age of Artificial Intelligence: How Artificial Intelligence can serve mathematical human learning*. Series: Mathematics Education in the Digital Era, Vol. 17, 2022, pp. 437-444. Springer Cham. https://rd.springer.com/content/pdf/bbm%3A978-3-030-86909-0%2F1.pdf

[8] Recio, T.; Vélez, P. (1999): Automatic Discovery of Theorems in Elementary Geometry. *Journal of Automated Reasoning* 23: 63-82.