PROBLEM CORNER

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Problem 1

The Fibonacci sequence

$$F_n = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

has fascinated hobbyists, scientists, professional mathematicians and many others. One of the many properties of these numbers is $F_n^2 + F_{n+1}^2 = F_{2(n+1)}$. Call this property **P**. The generating function for the Fibonacci numbers is

$$\frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_n x^n.$$

Lets experiment with *Mathematica*. Change the generating function to

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Mathematica produces the sequence

$$a_n = 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, \ldots$$

Examine the sequence to see that $a_n^2 + a_{n+1}^2 = a_{2(n+1)}$. The sequence also has property **P**. In fact, showing that the sequence has property **P** was Putnam problem A3, 1999. If you look up the sequence in the *Online Encyclopedia of Integer Sequences*, OEIS, you see that this is sequence A000129 the sequence of Pell numbers.
For this first problem experiment with *Mathematica* to determine if the sequences corresponding to the following generating functions

$$\frac{1}{1-2^2x-x^2} = \sum_{n=0}^{\infty} a_n x^n \qquad \frac{1}{1-2^3x-x^2} = \sum_{n=0}^{\infty} a_n x^n$$

$$\frac{1}{1-3x-x^2} = \sum_{n=0}^{\infty} a_n x^n \qquad \frac{1}{1-3^2x-x^2} = \sum_{n=0}^{\infty} a_n x^n$$

have property **P** and if the sequence appears in OEIS. Use *Mathematica* to find an explicit formula for a_n in the generating function $\frac{1}{1-3^2x-x^2} = \sum_{n=0}^{\infty} a_n x^n$.

Problem 2
The previous problem now provides motivation for your next challenge. Consider the generating function

$$\frac{1}{1-m^k x-x^2}=\sum_{n=0}^{\infty} a_n x^n$$

where m and k are integers with $m \geq 2$ and $k \geq 1$. Use technology such as *Mathematica* to show that

$$\begin{aligned} a_n^2+a_{n+1}^2 &= a_{2(n+1)} \\ &= \frac{\left(m^k+\sqrt{m^{2k}+4}\right)^{2n+3}-\left(m^k-\sqrt{m^{2k}+4}\right)^{2n+3}}{2^{2n+3}\sqrt{m^{2k}+4}}. \end{aligned}$$