Solutions to Problem Corner for February 2013

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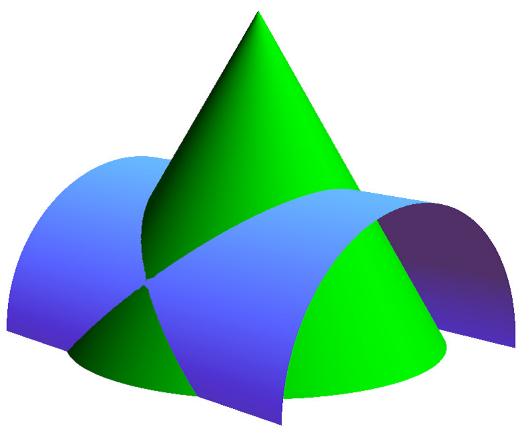
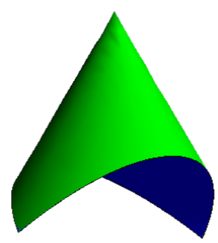
Question1&2. A solid cone has a base:$x^2 + y^2 \leq \frac{4}{3}$, $z = 0$ and a vertex A$(0, 0, 2)$, a solid cylinder with an equation: $y^2 + z^2 \leq 1$ is placed in R^3 as shown in the picture below. C is the intersection curve of the cone and the cylinder. Or C is the bottom edge of the green cone as shown in the right picture below:

1. What is the area of the region on the cylinder enclosed by the curve C?

2. What is the volume of the cone above the cylinder?

(i.e. $\boxed{1}$ is the region on the cylinder hidden inside the sharp cone as shown in the right picture. $\boxed{2}$ is the volume of solid bounded by the right green cone.)





Solution 1. m_θ is the line which passes a point $(0, \cos\theta, \sin\theta)$ on the surface of the cylinder and parallel to x axis. Then the equation of m_θ is;

$$m_\theta;\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \cdots (1)$$

The equation of the cone surface is;

$$3(x^2 + y^2) = (z - 2)^2 \qquad \cdots (2)$$

By combining equations (1) & (2), we get;

$$3(t^2 + \cos^2\theta) = (\sin\theta - 2)^2$$

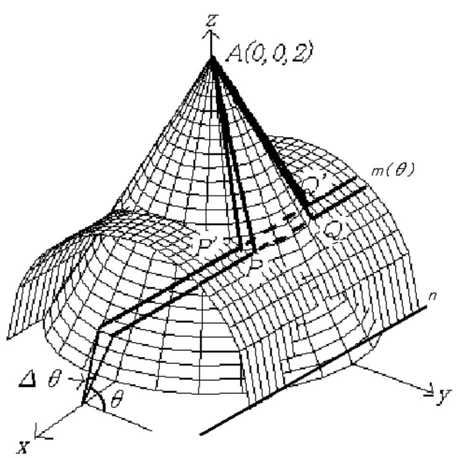
$$\begin{aligned} \therefore 3t^2 &= \sin^2\theta - 4\sin\theta + 4 - 3(1 - \sin^2\theta) \\ &= 4\sin^2\theta - 4\sin\theta + 1 \\ &= (2\sin\theta - 1)^2 \end{aligned}$$

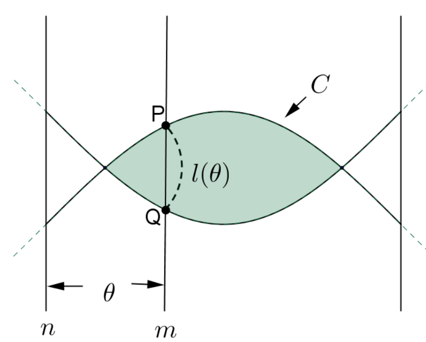
Thus, if we let P and Q be the intersection points of m_θ and the cone, and $l(\theta)$ be $\overline{PQ}$ (distance between P&Q), then

$$l(\theta) = 2|t| = \frac{2}{\sqrt{3}}|2\sin\theta - 1| \qquad \cdots (3)$$

Let n be a line: $y = 1, z = 0$. Then the angle of the rotation from m_θ to n around x axis is θ, thus if we develop the cylinder, the distance between m_θ and n is $1 \times \theta = \theta$. Because of that, the net of the cylinder ($z \geq 0$) is like in the right picture below. And the surface area is;

$$S = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} l(\theta) d\theta = \frac{2}{\sqrt{3}} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2\sin\theta - 1) d\theta = 4 - \frac{4\sqrt{3}}{9}\pi \qquad \cdots ans.$$





net of the cylinder

Note that in the integral above, we have used the bounday points $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. That is because these bounday points are on both surfaces of the cylinder and the cone. So if we plug in the point $(0, \cos\theta, \sin\theta)$ into equation $3(x^2 + y^2) = (z - 2)^2$, it gives us an equation about θ as $3\cos^2\theta = (2 - \sin\theta)^2$. We leave it to readers to verify that the solutions to this equation between 0 and π are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

Solution 2. The first half is the same as the problem **1**. P & Q are the intersections of m_θ and the cylinder. As in the problem **1**,

$$\overline{PQ} = \frac{2}{\sqrt{3}}|2\sin\theta - 1| \qquad \cdots(1)$$

Next, let α be the tangent plane to the cylinder at the line PQ. Then,

$$\alpha : (\cos\theta)y + (\sin\theta)z = 1 \qquad \cdots(2)$$

Let h be the distance between α and $A(0,0,2)$, then,

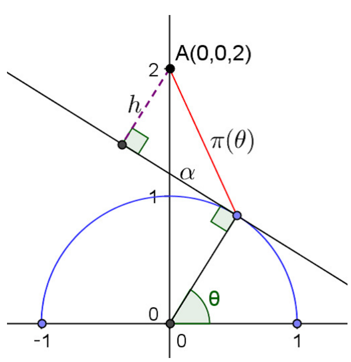
$$h = \frac{|2\sin\theta - 1|}{\sqrt{\cos^2\theta + \sin^2\theta}} = |2\sin\theta - 1| \qquad \cdots(3)$$

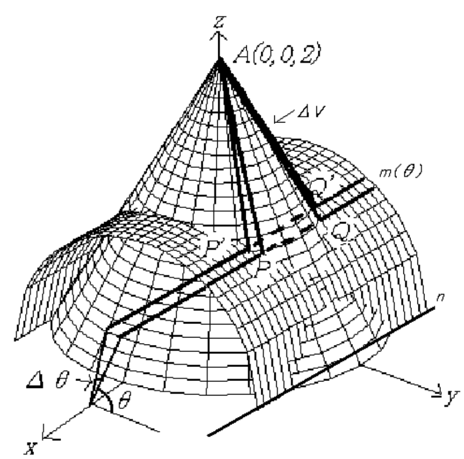
Let $\pi(\theta)$ be the plane which passes P, Q, and $A(0,0,2)$. Let $V(\theta)$ be the volume of "a part of the cone above the cylinder surrounded by $\pi\left(\frac{\pi}{6}\right)$ and $\pi(\theta)$", and let ΔV be the increments of V for $\Delta\theta$. Then ΔV is the volume of the quasi pyramid created by $\pi(\theta)$, $\pi(\theta+\Delta\theta)$, $\triangle AQQ'$, $\triangle APP'$ and the cylinder. Thus, when $\Delta\theta \approx 0$,

$$\Delta V \approx \frac{1}{3} \times (\square PQQ'P') \times h = \frac{1}{3} \times (PQ \cdot \Delta\theta) \times h = \frac{2}{3\sqrt{3}}(2\sin\theta - 1)^2 \Delta\theta \qquad \cdots(4)$$

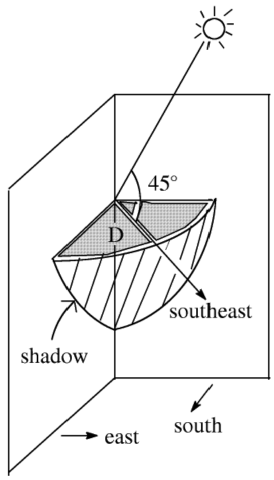
Therefore, again with the boundary points $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, V is

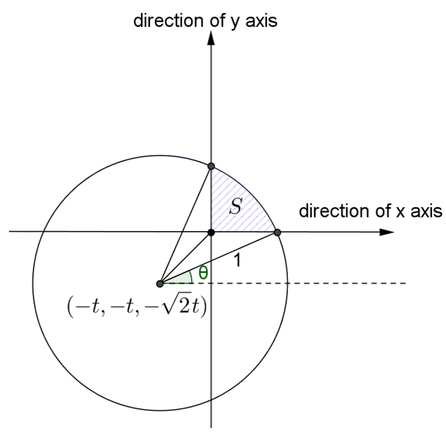
$$V = \frac{2}{3\sqrt{3}} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2\sin\theta - 1)^2 d\theta = \frac{4\sqrt{3}}{9}\pi - 2 \qquad \cdots ans.$$





Question 3. Suppose a shelf D with the right central angle and radius 1 is placed horizontally in a corner made of two walls; one wall faces the south and the other faces the east. And the sunlight comes from the southeast with 45 degrees against the ground. Suppose the shelf D is high enough (now shadow on the ground), what is the volume of the shadow of D created by the sunlight? (i.e. how many unit cubes can be hidden under the shadow?)





Solution 3. Take the origin O at the center of D, take x, y and z axes to the direction of the south, east and right above respectively. Then the direction vector of the sunlight is $(-1, -1, -\sqrt{2})$. Therefore **if there was no wall** the shadow of D is the translation of D in the direction of $(-1, -1, -\sqrt{2})$. Thus, the cross section of the shadow by $z = -\sqrt{2}t$ $(t \geq 0)$ is a shaded portion of a unit circle centered at $(-t, -t, -\sqrt{2}t)$. Take an angle $\theta \left(0 \leq \theta \leq \frac{\pi}{4}\right)$ as the right above, and let S be the area of the shaded portion, then,

$$t = \sin\theta,$$

$$z = -\sqrt{2}t = -\sqrt{2}\sin\theta,$$

$$S = \frac{1}{2}\cdot 1^2 \cdot 2\left(\frac{\pi}{4} - \theta\right) - \frac{1}{2}t(\cos\theta - t)\times 2 = \frac{\pi}{4} - \theta + t^2 - t\cos\theta,$$

$$\begin{aligned}
\therefore V &= \int_{-1}^{0}\left(\frac{\pi}{4} - \theta + \sin^2\theta - \sin\theta\cos\theta\right)dz \\
&= \sqrt{2}\int_{0}^{\frac{\pi}{4}}\left(\frac{\pi}{4} - \theta + \sin^2\theta - \sin\theta\cos\theta\right)\cos\theta d\theta \qquad \begin{array}{c|cc} z & -1 & \to 0 \\ \hline \theta & \frac{\pi}{4} & \to 0 \end{array} \\
&= \frac{\sqrt{2}\pi}{4}\int_{0}^{\frac{\pi}{4}}\cos\theta d\theta - \sqrt{2}\int_{0}^{\frac{\pi}{4}}\theta\cos\theta d\theta + \sqrt{2}\int_{0}^{\frac{\pi}{4}}\sin^2\theta\cos\theta d\theta - \sqrt{2}\int_{0}^{\frac{\pi}{4}}\sin\theta\cos^2\theta d\theta \\
&= \frac{\sqrt{2}\pi}{4}\Big[\sin\theta\Big]_{0}^{\frac{\pi}{4}} - \sqrt{2}\Big[\theta\sin\theta + \cos\theta\Big]_{0}^{\frac{\pi}{4}} + \frac{\sqrt{2}}{3}\Big[\sin^3\theta + \cos^3\theta\Big]_{0}^{\frac{\pi}{4}} \\
&= \frac{2\sqrt{2}}{3} - \frac{2}{3} \qquad \cdots ans.
\end{aligned}$$

* I came up with this problem when I saw a small shelf in my bathroom.

Using Mathematica to solve these problems can be found in this link.