PROBLEM CORNER

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Problem 1

Suppose that there are *n* people in a room. What is the exact probability that there are exactly two people (*A*,*B*) such that *A* and *B* have the same birthday and no others share a birthday? Note that this is a variation of the famous birthday paradox where the question is to determine the probability that at least two of *n* people have the same birthday. Determine the probability for n=5 as an exact fraction. Give a decimal approximation for n=30. Then give a formula for the general case for any $n>2$. Challenge: Extend the problem for two distinct pairs (*A*,*B*), (*C*,*D*) such that *A*,*B* share a birthday, and *C*,*D* share a different birthday and everyone else has a distinct birthday.

The picture above is from http://www.didyouknowblog.com. More interested readers can find an introduction to the "birthday paradox" problem from this website: https://en.wikipedia.org/wiki/Birthday_problem

Solution

The classic birthday paradox for *k* persons that, in a group of *n* people, the probability that there is at least one pair of people who have the same birthday is [1]

$$\mathrm{P}(at\ least\ one\ pair\ having\ the\ smae\ birthday|\, n\ people) = \prod_{i=1}^{k-1}\left(\frac{365-i}{365}\right)$$

For this problem we will show that the probability of at least two having the same birthday becomes greater than 1/2 when the room holds at least 23 persons. In Figure 1, the x-axis shows the number of people in the room, the vertical axis represents the probability that there are at least two people (*A*,*B*) such that *A* and *B* have the same birthday and no others share a birthday.

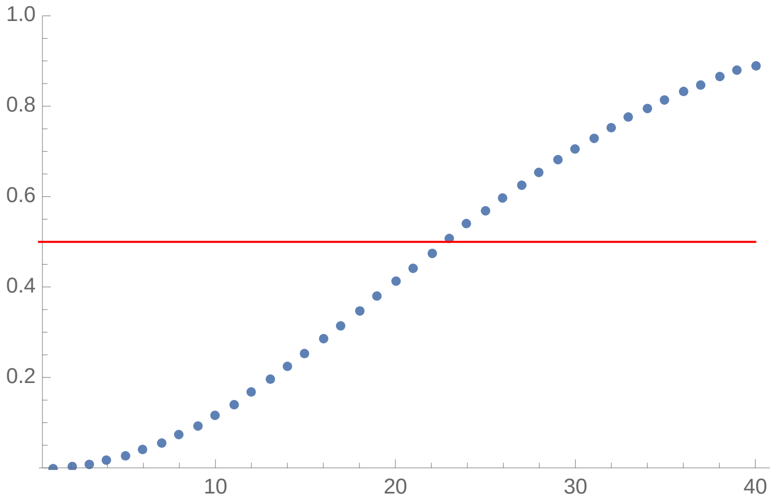


Figure 1

To solve the exact match problem we start with permutations. The number of permutations of *n* things taken *k* at a time is given by:

$$P(n,k) = \frac{n!}{(n-k)!}$$

We want to compute the number of different patterns of day choices (1-365) that contain exactly one date match. We do this as the product of three terms: (1) the number of ways to place the 2 matching elements in a list of *k* date choices, (2) the number of distinct patterns for the remaining *k*-2 choices that are all distinct from each other, and from the matching pair, and (3) the number of different matching day choices.

For example, if we have only 5 different values to choose from, and we choose 4 values. First, there are $\binom{4}{2}=\frac{4!}{2!2!}$ ways to position a matching pair in our list of four choices:

(1,2), (1,3), (1,4), (2,3), (2,4), (3,4),

Second, there are a total of $5^4 = 625$ different choices;

Finally, for each of these we have 5 different symbols that could have matched, *n*=5. For the remaining two symbols, we have 4 choices for the first, and 3 choices for the second, this is $4 \times 3 = 12$.

All of the discussions above give a total of $5 \times 6 \times 12$=360 choices that have exactly one match. This, out of the possible 625 choices, gives

$$P(exactly\ one\ pair|\ chosing\ 4\ people\ out\ of\ 5) = \frac{360}{625} = \frac{72}{125}$$

We generalize these ideas in the next function. The number of patterns of *n* things, taken *k* at a time with exactly one pair equal is given by the function:

$$Cnt(n,k) = n \times C(k,2) \times P(n-1,k-2),$$

where $C(k,2) = \binom{k}{2}$ is the number of ways to position a pair in a list of k elements;

$P(n-1,k-2) = \frac{(n-1)!}{(n-k+1)!}$ is the number of ways to choose k-2 distinct elements out of the rest n-1 elements. Note that for $P(n-1,k-2)$, the order matters because, for example, the day number choices 3, 17, 84, 17, 32 is counted as distinct from the choices 84, 17, 32, 17, 3 in our total of n^kpatterns of choice.

From the discussion above, the probability that choosing k persons out of n people, the probability that there is exactly one pair of two persons with the same birthday is

$$P(exactly\ one\ pair|\ chosing\ k\ people\ out\ of\ n) = \frac{Cnt(n,k)}{n^k}$$

where n^k represents the number of different choices of k elements out of n where duplicates are allowed. This is the same as the number of distinct k digit numbers base n.

For instance, using Mathematica, we can compute the exact probabilities for a pair with the same birthday for 5 people as $\frac{95663568}{3549780125}$. Similarly, the approximate probability of

exactly one pair out of 30 people is 0.380216.

Generally, we define P*(365, k) a*s

$$P(exactly\ one\ pair|\ out\ of\ k\ people) = P(match\ |k) = 365 \times \frac{k \times (k-1)}{2} \times \frac{364!}{(365-k)!}$$

A graph of Pr(365,k) for k=3 to 40 is shown in Figure 2. Notice that the probability never reaches 1/2, and declines after k=23 because the likelihood of more than one match grows quickly after that point.

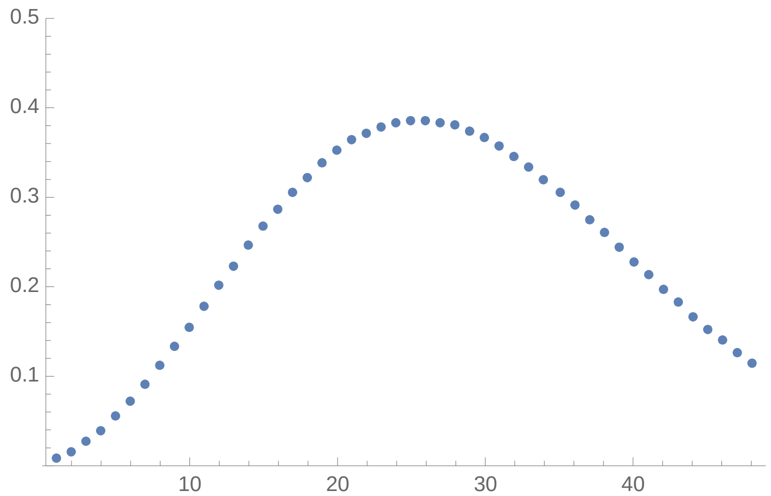


Figure 2

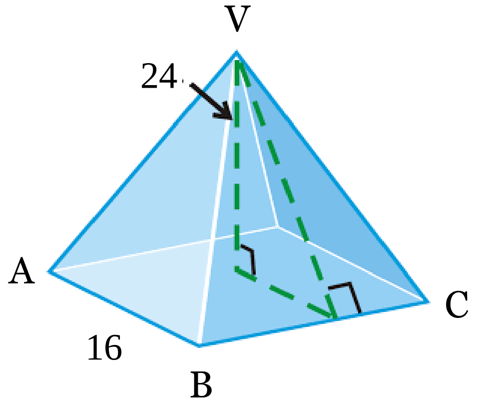
In the attached Mathematica notebook, we use a function defined as Pr(match | n,k) to represent the probability $P(exactly\ one\ pair|\ chosing\ k\ people\ out\ of\ n)$ as defined above. Then the value of Pr(match | 12,4)=$\frac{55}{144}$.

For the Birthday Paradox problem as in problem 1, we have P*(365, 18)=0.380216* as output from Mathematica.

Interested readers can find the Mathematica codes for simulation experiments for functions Pr(match | *n*, *k*) and P(365, *k*) at
here (put file 'eJMT_ProblemCorner_Problem01_February2016#1.nb').

Problem 2

Mr. Rich has purchased a piece of diamond in a right pyramid shape: the diamond has a square base ABCD of each side 16mm. Each sloping edge is 24mm long. Mr. Rich plans to embed this piece of diamond into the pendant of his wife's necklace so he needs to know the angle between the faces VAB and VBC. Can you help him? (The following picture is from "Surface Area and Volume of Pyramids" at http://www.ck12.org/geometry/Surface-Area-and-Volume-of-Pyramids/lesson/Pyramids/



Solution

The triangles in the problem are illustrated in Figure 3 below. We draw a height from vertex V on base face. Our objective is to determine the angle between adjacent faces. We will use the vector form of the law of cosines $\boldsymbol{u} \cdot \boldsymbol{v} = ||\boldsymbol{u}|| \times ||\boldsymbol{v}|| \times \cos(\theta)$. In order to do that, we construct a 3D coordinate system as following: Point F is the origin, vector $\overrightarrow{FE}$ is in the same direction of positive x-axis, vector $\overrightarrow{BC}$ is in the same direction of positive y-axis, and $\overrightarrow{FV}$ is in the same direction of positive z-axis.

To solve the problem we first determine the height $h = FV$. First note that the length BF is 1/2 of the diagonal of the base, so

$$BF = {16\sqrt{2}}/{2} = 8\sqrt{2}$$

Then by Pythagorean theorem, $h = \sqrt{24^2 - BF^2} = 8\sqrt{7}$, as illustrated in Figure 4 below.

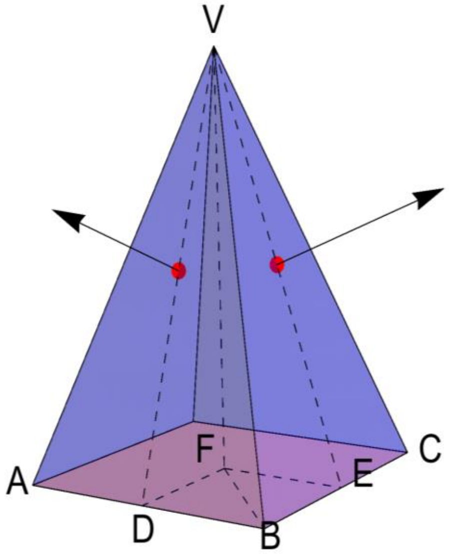


Figure 3

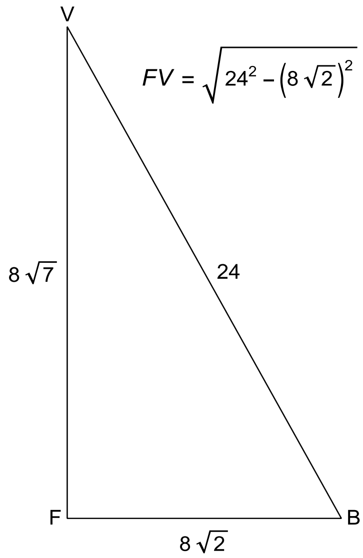


Figure 4

Next, we determine the sides of triangle ΔEFV, shown in Figure 5. The base FE is one half the length of a side of the pyramid base, so $FE = 8$. Then Pythagorous gives the length of the hypotenuse $EV = 16\sqrt{2}$. The vector $\vec{u}$ is perpendicular to the face VBC. Using the cross product of two vectors in plane VBC, we can get that the vector $\vec{u}$ is in the same direction of $< 8\sqrt{7}, 0, 8 >$. Similarly the vector $\vec{v}$, perpendicular to the adjacent face VAB is in the same direction of $< 0, -8\sqrt{7}, 8 >$. Now we use the formula of the dot product rule to calculate the angle between vectors $\vec{u}$ and $\boldsymbol{\vec{v}}$:

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\left||\vec{u}|\right| \times \left||\vec{v}|\right|} = \frac{64}{(16\sqrt{2})^2} = \frac{1}{8}$$

So the angle between vectors $\vec{u}$ and $\vec{v}$ is $cos^{-1}\left(\frac{1}{8}\right) = 82.8192^o$. Now the angle between the sides is $\pi - \theta$, as shown in Figure 5:

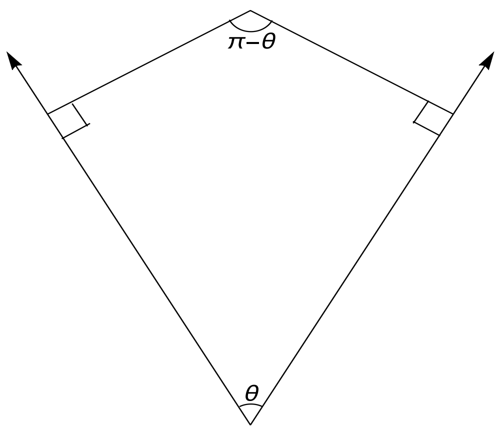


Figure 5

So in our case the angle between the planes interior to the diamond is

$$180^o - 82.8192^o = 97.1808^o.$$

Interested readers can find the Mathematica codes for the solution above can be found **here (put file 'eJMT_SpringProblem02_Solution #2 real')**.

Reference:

[1] "Birthday Problem", From Wikipedia, the free encyclopedia, https://en.wikipedia.org/wiki/Birthday_problem