

# PROBLEM CORNER

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1. In a remote, semi-barbaric kingdom<sup>1</sup>, the need for a strong army led the King to promulgate a cruel law: To raise the number of young men available, no family should have more than one daughter. Thus, every woman in the kingdom had a certain number of children, and the last of these (and *only* the last) was a girl. What was the final proportion of boys and girls? (Of course, you should assume that the events “having a child” are independent from each other, with a probability  $p = 1/2$  for a boy and  $p = 1/2$  for a girl).

*Solution:* According to the statement of the problem, each woman had children until reaching a girl, at which moment they stopped having children. Thus, if there were  $N$  women in the kingdom, exactly  $N_g = N$  girls were born. How about the boys? Half of the women had a girl first, so  $N/2$  women had 0 boys. Among the  $N/2$  that had boys, half of them had a girl as their next child, so  $N/4$  had exactly one boy. Iterating that reasoning,  $N/8$  had two boys, and generally  $N/2^{k+1}$  had  $k$  boys. Mathematically, we could write for  $N_b$ , the number of boys,

$$N_b = N \sum_{k=1}^{\infty} \frac{k}{2^{k+1}} = \frac{N}{4} \sum_{j=1}^{\infty} \frac{j}{2^{j-1}}.$$

On the other hand, it is well-known that, for  $|x| < 1$ ,

$$\sum_{j=1}^{\infty} jx^{j-1} = \frac{1}{(1-x)^2}$$

(just take the derivative of the formula for the sum of a geometric series).

Replacing  $x = 1/2$ ,

$$\sum_{j=1}^{\infty} \frac{j}{2^{j-1}} = \frac{1}{(1-\frac{1}{2})^2} = 4$$

and hence

$$N_b = \frac{N}{4} 4 = N.$$

As  $N_b = N_g$ , we see that the proportion of boys and girls was exactly the same that would have obtained by letting Mother Nature go her way.

2. Surely the reader has heard about “Flatland”, by E. Abbott, a book that describes the life of geometric creatures in the Euclidean plane. It is a somewhat boring world; for example, you can consider the fact that the only permitted motion in it, is along straight lines. Here is the proof: Given any smooth path  $x : I \subset \mathbb{R} \rightarrow \mathbb{R}^2 - \{0\}$ , let us see that there exists a unit vector  $u \in \mathbb{R}^2$  such that  $x(t) = |x(t)|u$ . To this end, notice that because  $x(t)$  does not take the value 0, we can compute (denoting by a point the derivative with respect to  $t$ )

$$\frac{d}{dt} \left( \frac{x(t)}{|x(t)|} \right) = \frac{|x|^2 \dot{x} - \langle x, \dot{x} \rangle x}{|x|^3},$$

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<sup>1</sup>Yes, this is a little homage to Stockton’s “The Lady or the Tiger”.

where  $\langle \cdot, \cdot \rangle$  denotes the scalar product in  $\mathbb{R}^2$ , and  $|\cdot|$  the Euclidean norm. Of course,  $\mathbb{R}^2 \subset \mathbb{R}^3$ , so we can make use of the identity

$$(u \times v) \times w = \langle u, w \rangle v - \langle v, w \rangle u,$$

which is valid for any vectors  $u, v, w \in \mathbb{R}^3$ . Thus,

$$\begin{aligned} \frac{d}{dt} \left( \frac{x(t)}{|x(t)|} \right) &= \frac{1}{|x|^3} ((x \times \dot{x}) \times x) \\ &= -\frac{1}{|x|^3} ((\dot{x} \times x) \times x) \\ &= -\frac{1}{|x|^3} (\dot{x} \times (x \times x)) \\ &= -\frac{1}{|x|^3} (\dot{x} \times 0) = 0, \end{aligned}$$

and there must be a constant vector  $u \in \mathbb{R}^2$  such that

$$\frac{x(t)}{|x(t)|} = u.$$

Is life in the plane really so boring?

*Solution:* Well, maybe the Euclidean plane is not the funniest place on Earth, but sure can not be that boring. The cross product is not associative, so the computation should have stopped after the second line.