PROBLEM CORNER

Wei-Chi YANG
wyang@radford.edu
Department of Mathematics and Statistics
Radford University, Radford, VA 24142

Example 1 *We are given a circle C with radius r_0 and centered at $O = (0, 0)$, and an ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, that is outside the given circle. Let A be a moving point on the circle. Suppose we construct the line OA to intersect at a point B on the ellipse. We construct the line l_1 passing through B and is parallel to $y - axis$. Next we construct the line l_2 passing through the point A and is parallel to $x - axis$. (a) Find the locus for the point P that is the intersection between l_1 and l_2. (b) Maximize the area of ABP.*

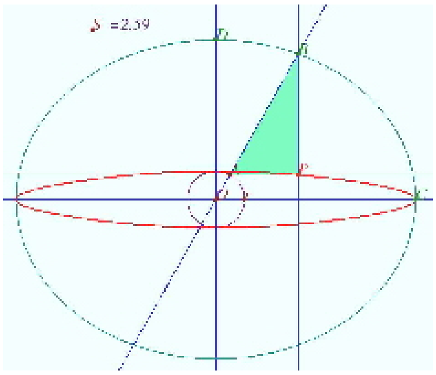

Figure 1 Generating the locus from a circle and an ellipse

We note that finding the locus for the point P can be solved by hand without too much work: We write $A = (A_x, A_y), B = (B_x, B_y)$, and let $OB = r, \measuredangle BOC = \theta$, then $B = (a\cos\theta, b\sin\theta)$. We see $OB^2 = a^2\cos^2\theta + b^2\sin^2\theta = a^2\cos^2\theta + b^2(1 - \cos^2\theta) = b^2 + (a^2 - b^2)\cos^2\theta$. Thus $r^2 = b^2 + (a^2 - b^2)\frac{1+\cos 2\theta}{2}$, which leads to

$$OB = r = \frac{\sqrt{2}ab}{\sqrt{a^2 + b^2 - (a^2 - b^2)\cos 2\theta}}. \tag{1}$$

If we write locus $P = (P_x, P_y)$, then $(P_x)^2 = \left(\frac{\sqrt{2}ab}{\sqrt{a^2+b^2-(a^2-b^2)\cos 2\theta}}\right)^2 \cos^2\theta = \frac{2a^2b^2\cos^2\theta}{a^2 + b^2 - (a^2 - b^2)\cos 2\theta}$ and $(P_y)^2 = r_0^2\sin^2\theta$. For part (b), the area of ABP is the absolute value of

$$\frac{1}{2}(AP)(BP) = \frac{1}{2}(P_x - A_x)(B_y - P_y) = \frac{1}{2}(r\cos\theta - r_0\cos\theta)(r\sin\theta - \sqrt{r_0}\sin\theta). \tag{2}$$

Now we substitute r in Eq. (10) into the area of ABP and use a CAS to simplify $\frac{1}{2}(AP)(BP)$ to the following form:

$$\frac{1}{4}\sin 2\theta\left(\frac{\sqrt{2}ab}{\sqrt{a^2+b^2-(a^2-b^2)\cos 2\theta}}-r_0\right)^2. \tag{3}$$

The locus corresponds to an ellipse is sketched in Figure 1 using *[2]*. If we use a CAS such as *[3]* with specific numeric values of $a=5, b=4$ and $r_0=\frac{1}{\sqrt{2}}$, we find the maximum area of ABP to be 3.5631, which occurs when θ is about 0.655308 radians or 37.5464 degrees.

Example 2 *We are given a circle C^* centered at $O=(0,0)$ with radius r_0, and a cardioid which resembles the shape of $r=a(1-\cos\theta)$, where $\theta\in[0,2\pi]$ enclosing the given circle C^* as shown in Figure 2(a). We are given a moving point A on the circle. Suppose we construct the line OA to intersect at a point B on the cardioid. We construct the line l_1 passing through B and is parallel to $y-axis$. Next we construct the line l_2 passing through the point A and is parallel to $x-axis$. Find the locus for the point P that is the intersection between l_1 and l_2.*

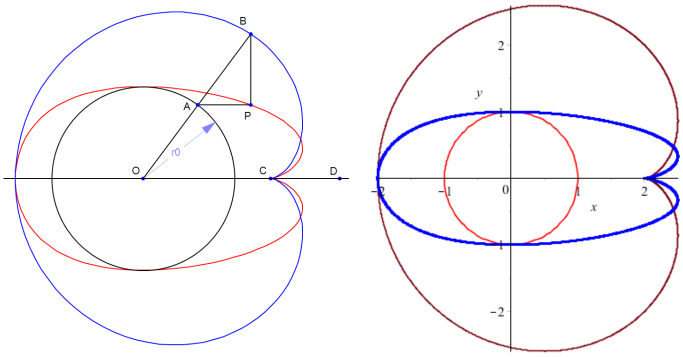

Figure 2(a) Locus generated by [1]

Figure 2(b). Locus generated by Maple [3].

First, we notice in Figure 2(a), the cardioid, of the shape $r=a(1-\cos\theta)$, enclosing the circle, is centered at the point C and the circle C^* is centered at $(0,0)$. If we use $O=(0,0)$ as the center of the cardioid enclosing the circle, we may write the parametric equation $[x(\theta),y(\theta)]$ for such cardioid as $x(\theta)=a(1-\cos\theta)\cos(\theta)+OC$ and $y(\theta)=a(1-\cos\theta)\sin(\theta)$ with $OC>r_0$. Now, we let $\theta=\measuredangle BOC, \varphi=\measuredangle BCD$, $OB=R, OC=a$. We next will express R in terms of a and angle φ. We write locus $P=(P_x,P_y)$, $A=(A_x,A_y)$, and $B=(B_x,B_y)$, and note $P_y=A_y=r_0\sin\theta$ and $P_x=B_x$. Note the original blue cardioid can be represented by $r=a(1-\cos\varphi)$. We observe $B_x=R\cos\theta=a+r\cos\varphi$ and $B_y=R\sin\theta=r\sin\varphi$, which leads to

$$\begin{aligned} R^2 &= a^2+2ar\cos\varphi+r^2 \\ &= a^2+2a\left(a(1-\cos\varphi)\right)\cos\varphi+a^2(1-\cos\varphi)^2 \\ &= a^2\left(2-\cos^2\varphi\right). \text{ This implies} \qquad (4) \\ R &= a\sqrt{2-\cos^2\varphi}. \qquad (5) \end{aligned}$$

In view of $P_x = B_x$, we see

$$\begin{aligned}\frac{P_x}{a} &= 1 + \frac{r}{a}\cos\varphi = 1 + (1-\cos\varphi)\cos\varphi = \sin^2\varphi + \cos\varphi, \\ P_x &= a\left(\sin^2\varphi + \cos\varphi\right). \qquad (6)\end{aligned}$$

Furthermore, we see

$$\begin{aligned}\frac{P_y}{r_0} &= \sin\theta = \frac{r}{R}\sin\varphi = \frac{r\sin\varphi}{a\sqrt{2-\cos^2\varphi}} = \frac{\sin\varphi\,(1-\cos\varphi)}{\sqrt{2-\cos^2\varphi}} \text{ and} \\ P_y &= r_0\left(\frac{\sin\varphi\,(1-\cos\varphi)}{\sqrt{2-\cos^2\varphi}}\right). \qquad (7)\end{aligned}$$

The parametric representative of locus P using angle φ is then $[P_x, P_y]$. We plot the locus $[P_x, P_y]$ together with cardioid and circle when $r_0 = 1$ and $OC = 2$ in Figure 2(b) with the help of *[3]*. If we make the substitution of $t = \tan\frac{\varphi}{2}$, then we can see that $t^4 + 4t^3\cot\theta - 4t^2 - 1 = 0$, which yields

$$\frac{P_x}{a} = \frac{2t^2}{1+t^2} \text{ and } \frac{P_y}{a} = \sin\theta. \qquad (8)$$

The Eqs. (6) and (7) represent the parametric equation for the locus P in terms of angle φ. The sketch of the locus corresponding to a cardioid is shown using *[1]* in Figure 2(a).

References

[1] Geometry Expression, see http://www.geometryexpressions.com/.

[2] Geometry in Mathematical Arts (GInMA): *A Dynamic Geometry System*, see http://deoma-cmd.ru/en/Products/Geometry/GInMA.aspx.

[3] Maple: A product of Maplesoft, see http://maplesoft.com/.

[4] Yang, W.-C. See Graphs. Find Equations. Myth or Reality? (pp. page 25-38). Proceedings of the 20th ATCM, the electronic copy can be found at this URL: http://atcm.mathandtech.org/EP2015/invited/2.pdf, ISBN:978-0-9821164-9-4 (hard copy), ISSN 1940-4204 (online version), Mathematics and Technology LLC.