PROBLEM CORNER

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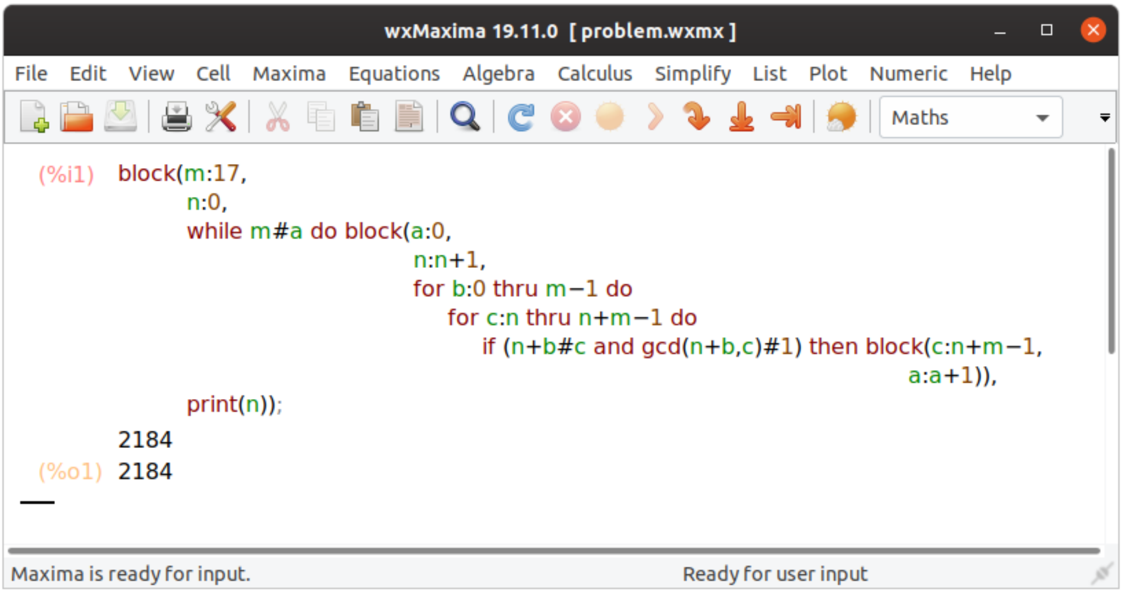
Problem 1

S. S. Pillai proved in [1] the following theorem. When $m \leq 16$, in every set of m consecutive integers, there is at least one integer which is relatively prime to all the others in the set. He realized that the smallest number m for which there is a sequence of m consecutive integers without a number relatively prime to all others is $m = 17$. Find the smallest natural number n such that in the sequence $n, n + 1, \ldots, n + 16$ there is no number relatively prime to the others.

[1] S.S. Pillai, *On m consecutive integers-I.*, Proceedings of the Indian Academy of Sciences Vol 11/1, pp 6-12 (1940).
https://www.ias.ac.in/article/fulltext/seca/011/01/0006-0012

SOLUTION

The answer is $n = 2184$. We can find it using a program, like this Maxima worksheet. (It is in wxMaxima environment.)



Problem 2

Draw all the diagonals of a regular polygon. How many interior intersections are there on a given diagonal? For example, the figure shows a regular 18-gon and all its diagonals. The diagonal connecting the opposite vertices has 27 interior intersection points.

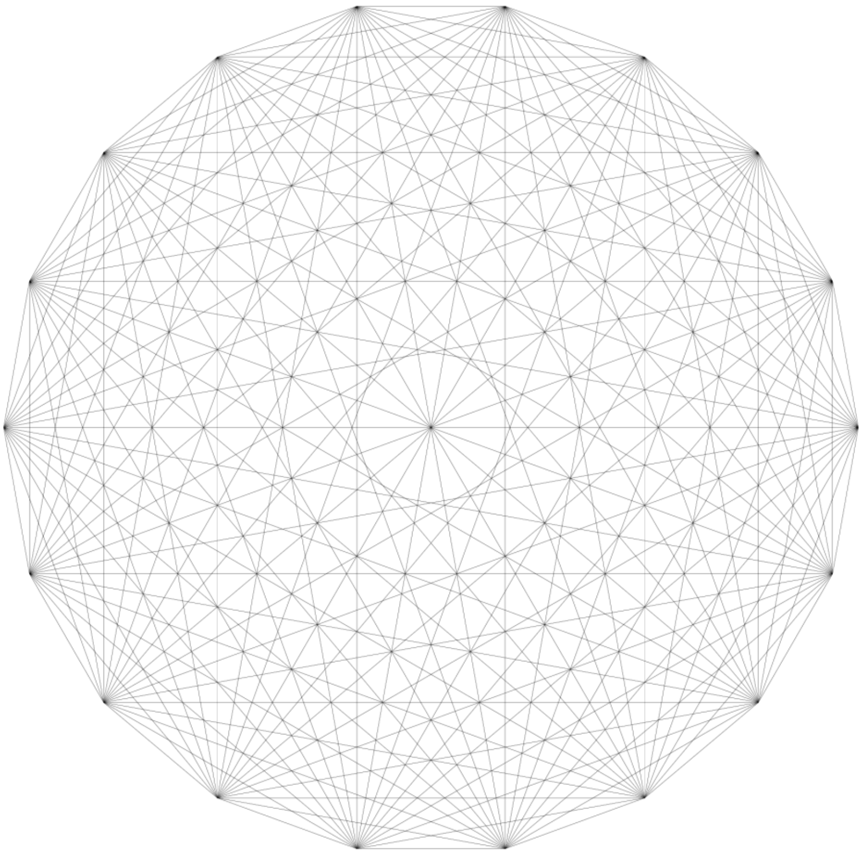


Figure. The regular 18-gon with its diagonals. There are 27 interior intersection points on the longest diagonal: 8 two-line intersections, 12 three-line intersections, 6 five-line intersections, and 1 nine-line intersection

Solution

Give the vertices of the regular n-gon with a serial number counterclockwise after the vertex 0 is fixed (Figure 1). A diagonal can thus be defined by two numbers. Fix the diagonal (0, i) and intersect it with the diagonal (j, k) as shown in the figure. Write the law of sines in the triangles BMC and AMB:

$$\frac{x}{z} = \frac{\sin(n-k)\alpha}{\sin(i-j)\alpha} = \frac{\sin k\alpha}{\sin(i-j)\alpha}, \frac{z}{y} = \frac{\sin(k-i)\alpha}{\sin j\alpha},$$

where $\alpha = \frac{\pi}{n}$. From this we can conclude that

$$\frac{x}{y} = \frac{\sin k\alpha}{\sin(i-j)\alpha} \cdot \frac{\sin(k-i)\alpha}{\sin j\alpha}.$$

If we fix i, then j=1, 2, …, i-1, and k=i+1, i+2, …, n-1. It means that the maximum number of interior intersection points is $(i-1)\cdot(n-i-1)$. However, we may have multiple intersection points. Then, we need to test whether different diagonals (j, k) and (j ', k') produce the same ratio or not. If not, we have a double intersection point (i.e. (j, k) intersects (0, i)). If two diagonals produce the same ratio, then the intersection is triple (i.e. (j, k) and (j', k') intersects (0, i) at the same point), and so on.

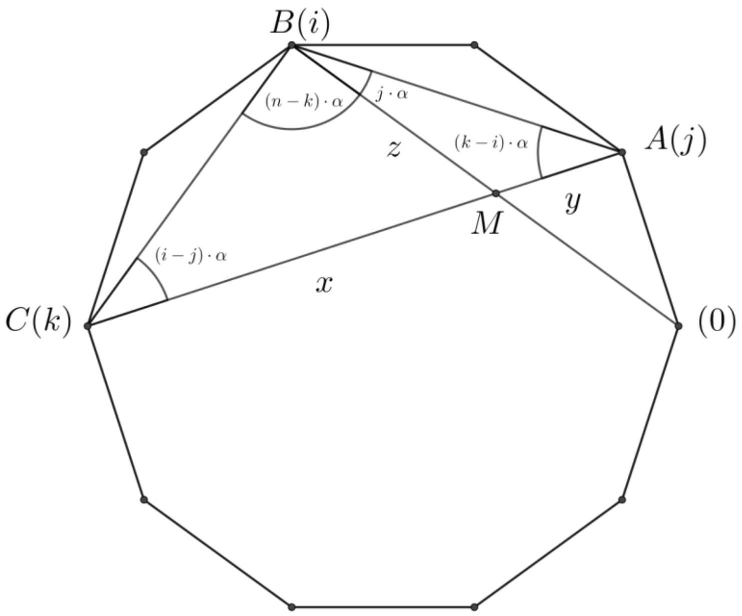


Figure 1.

A Wolfram Cloud implementation is here:

https://www.wolframcloud.com/obj/kovacs764/Published/diagonals.nb

The password is ProblemCorner2020.