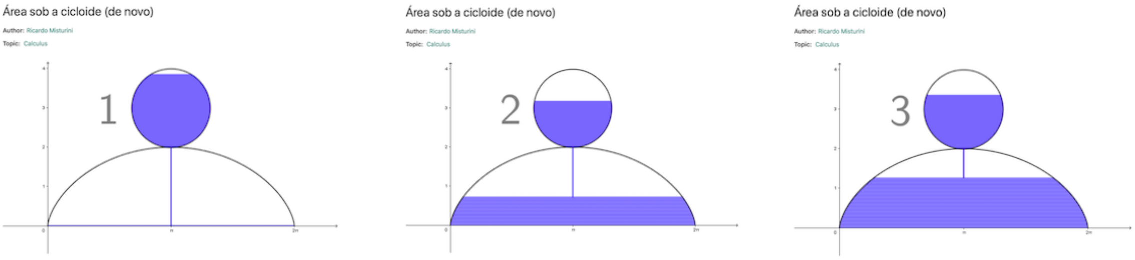
PROBLEM CORNER

Provided by Eduardo Toffolo[1], Diego C. Nunes[2] and Diego Lieban[2]
[1]UNICAMP - Universidade de Campinas
[2]IFRS – Instituto Federal do Rio Grande do Sul
E-mail: edu.oloffot@gmail.com and diego.lieban@bento.ifrs.edu.br

MOTIVATION

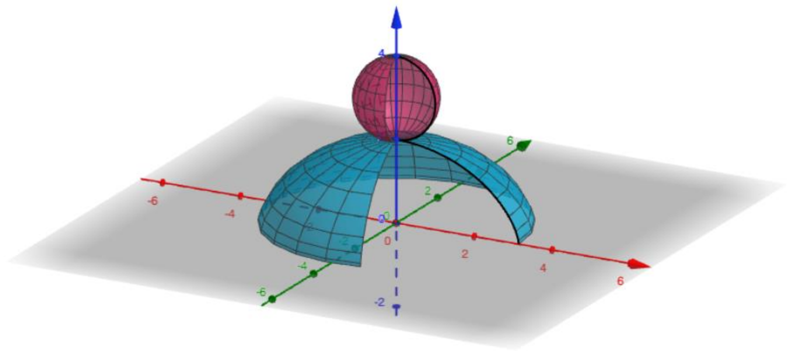
Cycloid is a well-known curve generated by the trace of a point on a circle, which rolls along a straight line without slipping or gaps. Historically it was already studied by several mathematicians and based on such contributions, it is nowadays a popular and important curve due to its physical interpretations. It is also quite common in Calculus courses, as a typical Integration exercise, to show that the area under one arc of this curve is equal to 3 times the area of the correspondent generator circle. A nice applet on GeoGebra Platform to illustrate (not proving) this fact is shared bellow:



https://www.geogebra.org/m/hacg6ex6

PROBLEM 1

Is the same relation still valid when we consider a 3D extended version of this? That means, does this relation remain for the volume under the revolution of the cycloid around its axis of symmetry and the volume of the corresponding sphere[1]?



HINT: For those who are not familiar with Calculus, it might help to consider the revolution of a right triangle, which has its shorter sides equal to the height of the cycloid and half the period of the cycloid (2 and π, respectively, when we consider the radius of the original circle equal to 1). The GeoGebra applet in the link below may be useful.

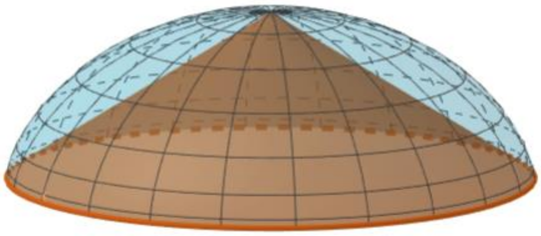
https://www.geogebra.org/m/tykauqhp

Solution:

The volume can be estimated by inscribing polygons on the cycloid and rotating them. The simplest case is the triangle and, turning it, a cone is obtained whose radius at the base measures $r \cdot \pi$, where r is the radius of the circle that generated the cycloid and the height is $2r$.

[1] Sphere generated by a complete revolution of half of the circle which originally generated the cycloid.

Therefore, the volume of this solid is $\frac{2\pi^3 r^3}{3}$, which means it is about 4.9 times greater than the volume of the sphere of radius r. Since the cone is entirely inside of the revolution of the cycloid, the volume of revolution of the cycloid is greater than the volume of the cone, i.e., the answer to the first question is “no”.



PROBLEM 2

Find (and prove) the ratio between the volume of the solid generated by the rotation of the cycloid around its axis of symmetry and the volume of the corresponding sphere.

Solution:

To find the exact ratio between the volumes of the solid generated by the rotation of the cycloid and the corresponding sphere, we appeal to Calculus (*Shell Method*). The cycloid whose axis of symmetry is the *z axis* can be described, parametrically, by

$$x = f(t) = r(t - \sin(t) - \pi)$$
$$y = 0$$
$$z = g(t) = r(1 - \cos(t)),$$

where $0 \leq t \leq 2\pi$. Let F be the function such that $z = F(x)$. The volume of the revolution of the cycloid around the *z axis* is then given by

$$V = 2\pi \int_0^{\pi r} x \cdot F(x)\, dx.$$

Replacing x by $f(t)$ (note that $g(t) = F\big(f(t)\big)$ and $f'(t) = g(t)$, the integral will be

$$2\pi \int_{\pi}^{2\pi} f(t) \cdot g^2(t)\, dx = 2\pi r^3 \int_{\pi}^{2\pi} (t - \sin(t) - \pi) \cdot (1 - \cos(t))^2 dt.$$

Expanding and calculating the integral, we obtain that the volume is

$$V = 2\left(\frac{3\pi^2}{4} - \frac{4}{3}\right)\pi r^3.$$

Therefore, the ratio between the volumes of the solid generated by the rotation of the cycloid and the corresponding sphere is

$$\frac{3}{2}\left(\frac{3\pi^2}{4} - \frac{4}{3}\right).$$

NOTE

This problem aims to enhance the 3D reasoning with GeoGebra based on Integral principles. Despite the techniques suggested for the solutions are slightly different, both of them use the idea of infinite sums. An alternative and interactive applet to explore the *Shell Method* used in Problem 2 can be found in https://www.geogebra.org/m/ggfye7dj.