PROBLEM CORNER

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PROBLEM 1

The book (Lam & Pope) attributes the following pentagon construction from an A4 paper to David Collier, which it notes gives an approximately regular pentagon (Figure 1).

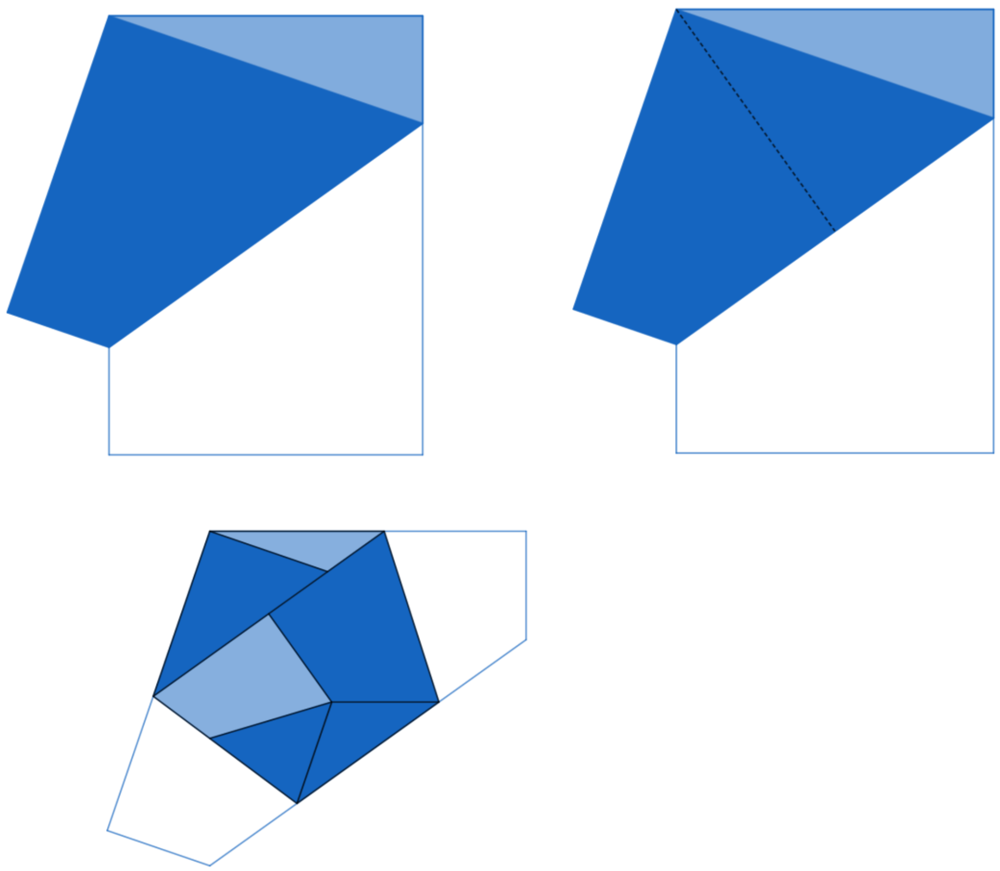

Figure 1. Pentagon construction

What should be the ratio of the sides of the rectangle so that the resulting pentagon is a regular pentagon?

Reference

Lam, T. K., & Pope, S. *Learning Mathematics with Origami.* Association of Teachers of Mathematics.

SOLUTION

First, fold out the sheet of paper. The third- and fourth-fold lines appear as open polygons on the sheet (Figure 2). Next, construct the pentagon using GeoGebra, which will give ideas to answer the question (Figure 3). The first fold line is the bisector of the DB diagonal of the rectangle. The second is the diagonal of the rectangle, i.e., the symmetry ax of the pentagon.

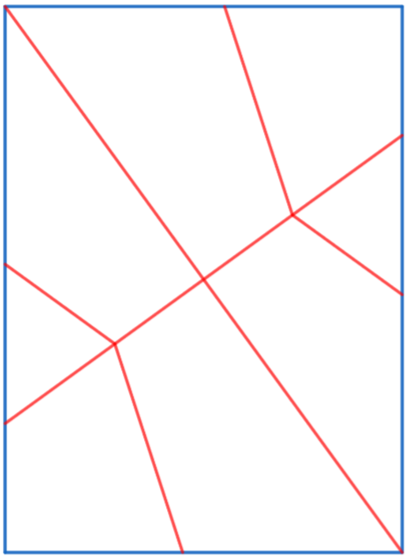

Figure 2. The folded-out sheet of paper

If GeoGebra is used to construct the third and the fourth fold lines (see https://www.geogebra.org/m/gguvq7yr) , the angle bisector of the DBC angle and the DBA angle must be inserted (f and g). The pentagon is immediately visible on the GeoGebra drawing area ($PQDQ'P'$).

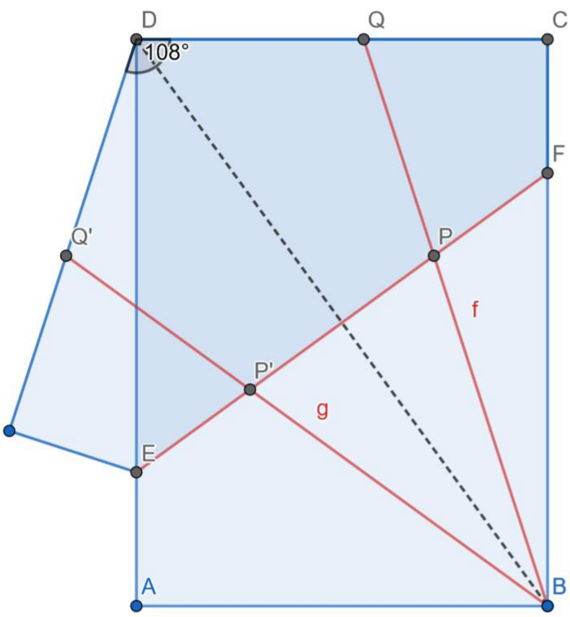

Figure 3. Construction with GeoGebra

We see that for the pentagon to be regular, the BDQ angle must be 54°. So the aspect ratio in question should be $1{:}\tan 54°$. However, we still need to prove that the result is a regular pentagon if we start from this aspect ratio sheet. The angle $DBQ\angle = 36^{\circ}/2 = 18^{\circ}$ and from this fact, it follows that all the angles of the pentagon equal 108° (Figure 4).

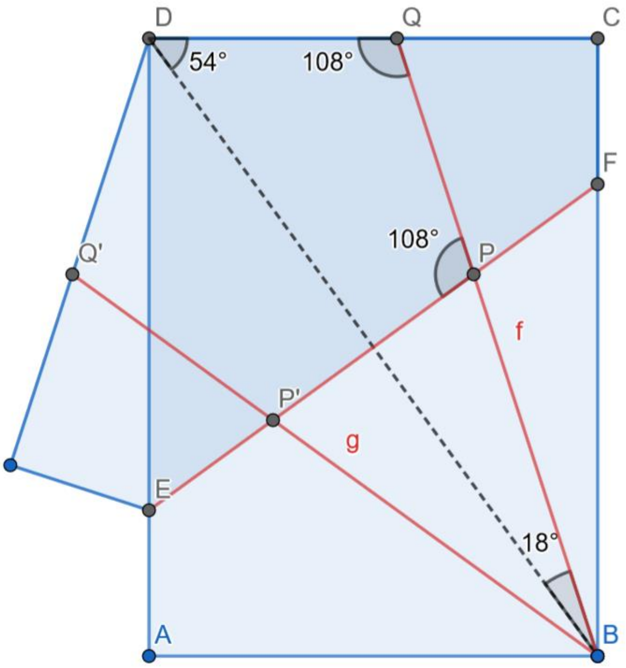

Figure 4. All the angles equal 108°

The sides of the pentagon are as follows:

$$PP' = tan(18^{\circ})\sqrt{1 + tan^2(54^{\circ})}\,,\; DQ = 1 - tan(54^{\circ})\,tan(18^{\circ}).$$

Computing the values with WolframOne, we get

```
In[1]:= 1 - Tan[18 °] * Tan[54 °]
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Out[1]= $1-\sqrt{\left(1-\frac{2}{\sqrt{5}}\right)\left(1+\frac{2}{\sqrt{5}}\right)}$

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In[2]:= Tan[18 °] * Sqrt[1 + Tan[54 °] ^ 2 ]
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Out[2]= $\sqrt{\left(1-\frac{2}{\sqrt{5}}\right)\left(2+\frac{2}{\sqrt{5}}\right)}$

It is easy to see that the expressions above are equal.

Now, we have an equiangular pentagon with the property that three sides (DQ, DQ', PP') are equal, and the other two sides are also equal: $PQ = P'Q' = a$. Let z be the complex number parallel with the vector PQ on the Gauss-plane, and $|z| = 1$ where $P'P$ is horizontal and of unit length (Figure 5).

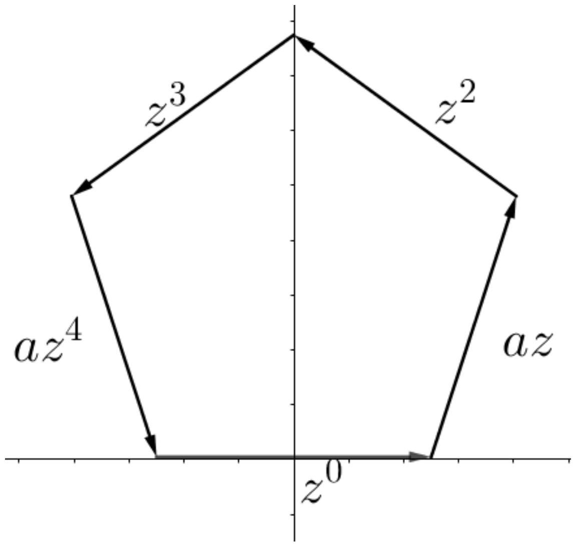


Figure 5. The pentagon on the complex plane

Then

$$z^0 + az^1 + z^2 + z^3 + az^4 = 0.$$

Moreover

$$z^0 + z^1 + z^2 + z^3 + z^4 = 0.$$

Subtracting the above equations, we get

$$(a-1)z^1 + (a-1)z^4 = 0,$$

Which fact implies that $a = 1$, and all the sides of the pentagon are equal.

PROBLEM 2

MOTIVATION

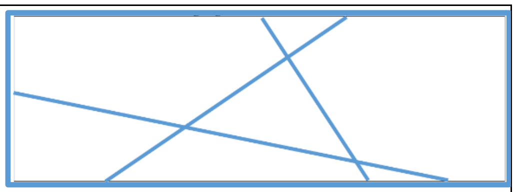
There is a famous problem, sometimes referred to as the "water well problem" (Goddijn & Reuter, 1995; Büchter & Leuders, 2005, p. 33), dates back to Descartes and Dirichlet who stated the problem in 1644 and 1850, respectively: "How can a plane be divided into areas (polygons) so that each point in an area is closer to the generating point than to any other generating point?" (Fisher, 2004)

This problem has been used in didactical settings to introduce the idea of perpendicular bisectors in problem-oriented teaching (e.g., Holzäpfel et al., 2016; Möller & Rott, 2017).

PROBLEM AND VARIATIONS

Here, we want to pose a similar yet quite different problem, to the water well problem described above. You still want to reach water as quickly as possible:

The canal problem
The map shows a parcel of land. There are three canals in this area.
Develop a partition of the areas into regions in a way that from each place in a region the canal in that region is the closest.



The canal problem was derived from the water well problem as a variation, by slightly altering the conditions of the problem (the distance to the lines instead of points in this case) (cf. Brown & Walter, 1983; Silver, 1994). Can you pose an easier or a more difficult problem by varying the canal problem? Does the solution strategy change when there are fewer or more canals? What if there were (curved) rivers instead of (straight) canals? What if there were lakes (that do or do not look like circles)?

References

Büchter, A. & Leuders, T. (2005). *Mathematikaufgaben selbst entwickeln*. Berlin: Cornelsen Verlag Scriptor.

Brown, S. I., & Walter, M. I. (1983). *The art of problem posing*. Franklin Institute Press.

Fisher, J. (2004). Visualizing the Connection Among Convex Hull, Voronoi Diagram and Delaunay Triangulation. Unpublished paper. https://pages.mtu.edu/~shene/PUBLICATIONS/2004/Hull2VD.pdf

Goddijn, A. J. & Reiter, W. (1995). Afstanden, grenzen en gebiedsindelingen (Nieuwe wiskunde tweede fase). Utrecht: Freudenthal Instituut.

Holzäpfel, L., Rott, B., & Dreher, U. (2016). Exploring perpendicular bisectors: The water well problem. In A. Kuzle, B. Rott, & T. Hodnik Čadež (Eds.), *Problem Solving in the Mathematics Classroom – Perspectives and Practices from Different Countries* (pp. 117–130). Münster: WTM.

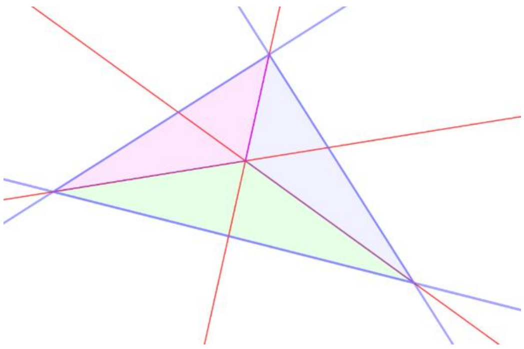
Möller, A. & Rott, B. (2017). Können durch problemorientierten Unterricht in derselben Unterrichtszeit vergleichbare Schülerleistungen erzielt werden? In U. Kortenkamp & A. Kuzle (Eds.), *Beiträge zum Mathematikunterricht 2017* (pp. 673–676). Münster: WTM.

Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics*, *14*(1), 19–28.

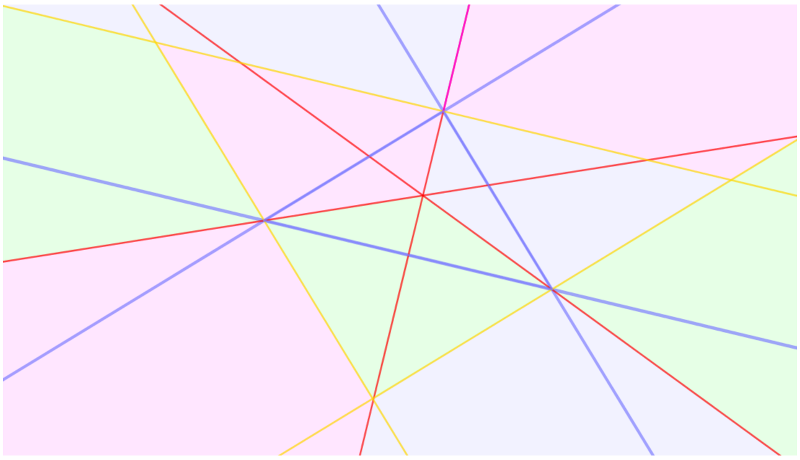
SOLUTION

The water well problem can be solved by using perpendicular bisectors, i.e. lines that have the property that each of their points is equidistant to two given points which are water wells in this context. Constructing a perpendicular bisector for each pair of water wells results in a so-called *Voronoi diagram* (see Fisher, 2004).

For the canal problem, borders are needed that are equidistant to lines. As no pair of canals is parallel to each other, angle bisectors are what is needed. A solution for the area that is enclosed by the three canals could look like this (with three canals, we have a triangle and, thus, construct the center of the incircle):



Wait, there are more areas outside the triangle that also need to be partitioned. To do this, we look at the parts of the angle bisectors that lie outside the triangle; and we also need to construct additional angle bisectors at the corners of the triangle for the pairs of angles that are outside of the triangle. We now have six angle bisectors that divide the whole area into polygons and for each polygon we need to decide which canal is the closest one. This leads to a partition of the whole area that looks like this:



The solution to the canal problem is also a Voronoi diagram, which is also true for the solutions of the further variations hinted at in the problem corner. The interested reader can learn more about such diagrams with curved boundaries in Ramamurthy and Farouki (1999a, 1999b).

References

Fisher, J. (2004). Visualizing the Connection Among Convex Hull, Voronoi Diagram and Delaunay Triangulation. Unpublished paper. https://pages.mtu.edu/~shene/PUBLICATIONS/2004/Hull2VD.pdf

Ramamurthy, R., & Farouki, R. T. (1999a). Voronoi diagram and medial axis algorithm for planar domains with curved boundaries I. Theoretical foundations. *Journal of Computational and Applied Mathematics, 102*, 119–141.

Ramamurthy, R., & Farouki, R. T. (1999b). Voronoi diagram and medial axis algorithm for planar domains with curved boundaries — II. Detailed algorithm description. *Journal of Computational and Applied Mathematics*, *102*, 253–277.