PROBLEM CORNER

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MOTIVATION

It is well known that Dynamic Geometry software is an excellent tool for teaching and learning geometry. See, for example, [4] and other chapters from the same volume where it is included. But in this Problem Corner we will not deal with problems to be solved by humans using Dynamic Geometry, but with *problems to be solved by GeoGebra automated reasoning tools*, available either at the standard GeoGebra version (www.geogebra.org) or in an experimental fork: *GeoGebra Discovery*, available in two options: GeoGebra Classic 5, (the one used in the following, *GeoGebra Discovery* version 2022May04, based on GeoGebra Classic 5.0.641.0-d), for Windows, Mac and Linux systems, that can be downloaded from https://github.com/kovzol/geogebra-discovery; and GeoGebra Classic 6, made for starting it in a browser at http://autgeo.online/geogebra-discovery/, mainly ready for use on tablets and smartphones. Details about the different available automated reasoning tools (*Relation*, *Prove*, *Discover*, *Compare*, *LocusEquation* commands, etc) can be found at [5], [6].

Thus, *the challenge here is for humans to help GeoGebra to solve the proposed problems* [7]. The context of both is the following fact: GeoGebra deals mostly with geometric statements that can be translated to algebraic equations (i.e., *not* involving inequalities). Although it is already possible to handle some inequalities (see [2]) it is on-going work to fully extend GeoGebra proving tools in this direction, given the high complexity (required amounts of memory and time) of such generalization. Thus, currently, we must think of some alternatives to approach, through GeoGebra, the proof of statements that include, for example, the bisector of an angle defined by two lines, as it is not possible to distinguish, without using inequalities, between the two possible bisectors associated to the two lines. This is the underlying issue concerning next Problem 1. A similar, even more involved, situation comes concerning Problem 2, where GeoGebra is faced with an optimization problem, where inequalities are implicit.

PROBLEM 1

Let I, O, H, denote the incenter, circumcenter and orthocenter of triangle ΔABC, respectively. Find necessary and sufficient conditions for the alignment of the three points.

We make the basic construction with GeoGebra (see Figure 1). Thus, f is the perpendicular bisector of AB, C is a point on f (so the triangle ΔABC is isosceles and side a= side b). Now consider g, the perpendicular bisector of side a, and let O be the circumcenter, i.e.

the intersection of f and g. Then build the line k through B perpendicular to AC, and let H be the orthocenter, as the intersection of f and k. Finally, consider lines l (bisector of angle(CBA)) and m (bisector of angle(BAC), and their intersection at the incenter I.

If we ask GeoGebra to find the relation between I and the Line(O, H), i.e. Euler's line, GeoGebra is unable to give a rigorous, affirmative assertion, only a numerically approximate answer.

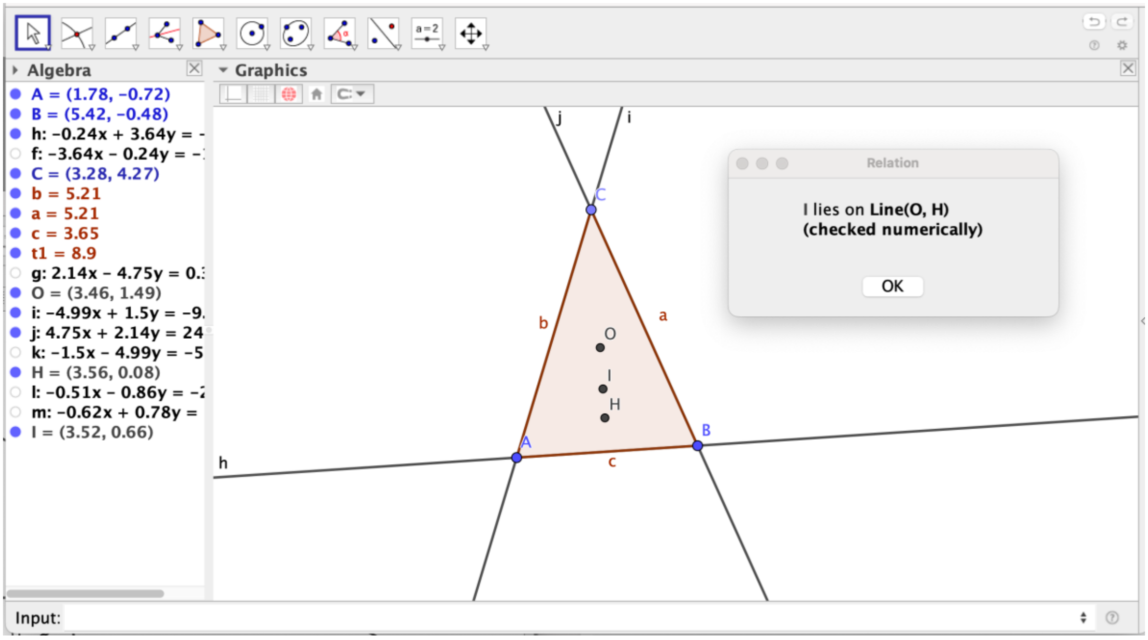

Figure 1: Given an isosceles triangle (a=b), O=circumcenter, H=orthocenter, I=incenter, *Relation*(I, Line (O,H)) does only answer approximately that I belongs to the Euler line.

It must be remarked that there is an option (in *GeoGebra Discovery*) to build directly the incenter I using the *IncircleCenter* command (that defines the incenter as the center of the circle tangent to the three sides of the triangle), but the answer to the *Relation*(I, Line (O,H)) is also only numerical in this case, as *IncircleCenter* is still under development and there are several circles tangent to the sides of the triangle: restricting the definition to the incircle that lies inside the triangle requires, again, to deal with inequalities. See [8] and [3] for a similar approach and related difficulties.

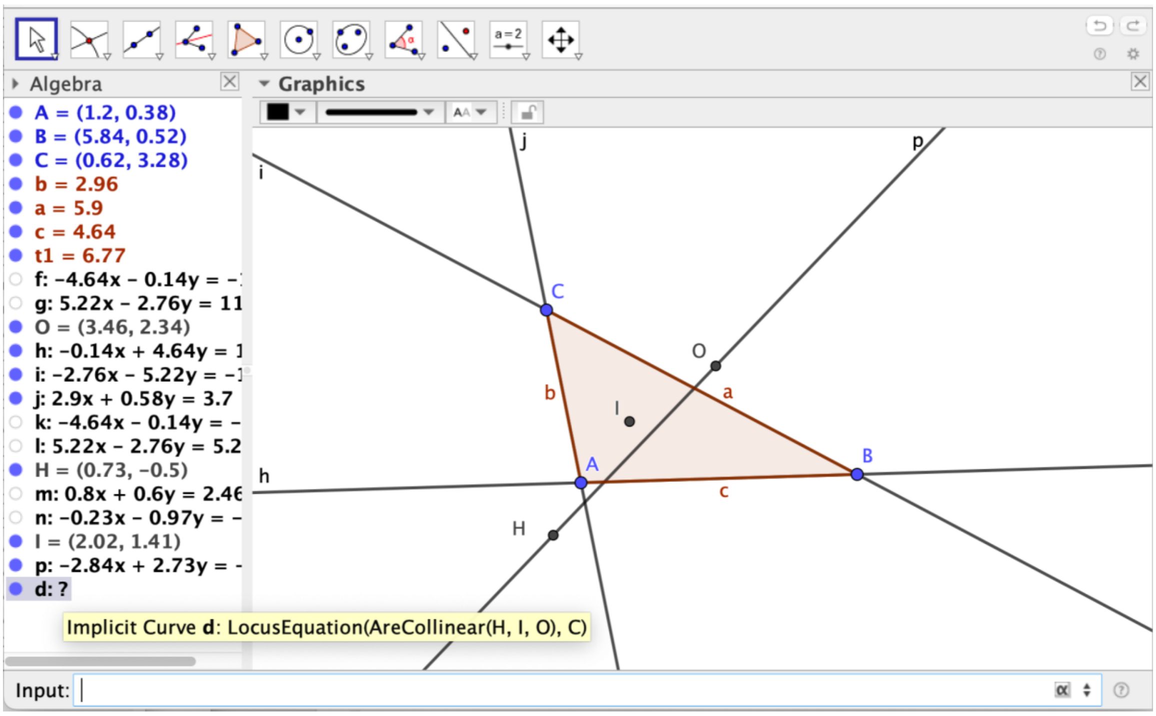

Figure 2: Conversely: LocusEquation of C for AreCollinear (H, I, O) yields "?".

Conversely, now we start with an arbitrary triangle ΔABC and we build, as the intersection of angle bisector lines, the Incenter I, the Orthocenter H and the Circumcenter O. Let p be the Euler line OH. Then we want to prove (see Figure 2) that if I lies on the Line (O, H), then the triangle must be isosceles, but GeoGebra ignores the locus of C for the triangle ΔABC to verify the collinearity of I, O, H.

PROBLEM: find an alternate way to deal with incenters that do not rely on signs, so that GeoGebra is able to find necessary and sufficient conditions for the alignment of the I, O, H.

HINT: Instead of starting with a triangle and then constructing the incenter, start with a given incenter and build the triangles having such incenter. Try, for the *LocusEquation* issue (Figure 2), to see if the *IncircleCenter* command helps at all.

Solution to Problem 1.

1.1 Classic solution

Let I, O, H, denote the incenter, circumcenter and orthocenter of ΔABC, respectively.

First, we will show that if ΔABC is isosceles, then I − O − H are collinear. WLOG, let CA= CB, as in Figure 1. It is enough to note that I, O and H would all belong to the line through C, perpendicular to AB.

Now, we will show that if I − O − H are collinear, then ΔABC isosceles. For this, we will use the following Lemma:

Lemma: In any triangle, its orthocenter and circumcenter are isogonal conjugates.

Let us recall that two lines r, s, are isogonal with respect to an angle $\angle XYZ$ if and only if r and s are symmetrical with respect to the interior bisector line of $\angle XYZ$.

In fact, in any non-necessarily isosceles triangle ΔABC, AO and AH are isogonal, because, if we let $\measuredangle OAC = \alpha$, then it holds that $\measuredangle ACO = \alpha \Longrightarrow \measuredangle COA = 180o - 2\alpha \Longrightarrow \measuredangle CBA = 90o - \alpha \Longrightarrow \measuredangle BAH = \alpha$ Similarly, BH and BO are isogonal, and so must be CH and CO, implying that the Lemma is true.

Now, suppose $I - O - H$ were collinear and that ΔABC wasn't isosceles. Then, in light of the Lemma, we get that I is the foot of the bisector of angles A, B, C, in triangles ΔAHO, ΔBHO, ΔCHO, respectively. Hence, in view of the Bisector Theorem we get that

$$HI / IO = AH/AO = BH/BO= CH/CO$$

By the definition of the circumcenter, it must hold $AO = BO = CO$, and so $AH = BH = CH$.
However, this implies that $H \equiv O$. Then, AH, BH, CH must be the perpendicular bisector lines of segments BC, CA, AB, respectively, and so $AB = BC = CA$.

1.2 Solving with GeoGebra

We start building a general triangle starting with two vertices A, B, in the line f, and then we consider two auxiliary points Aux1, Aux2, so that the first one defines the bisector line g of angle CBA and the second defines the bisector line h of angle BAC. See Figure 3. Then we reflect line f with respect to line g, getting line f'. Likewise, we reflect f with respect to line h, getting line f'_1. Both lines intersect at the third vertex of the triangle, C. By construction, the intersection of h and g is the incenter I. Notice that, if we build the third bisector i (of angle ACB), we expect lines i, g, h to be concurrent, but the output of the GeoGebra Prove command is "undefined", since a positioning Aux1 and Aux2 in other positions (e.g. one of them below line f, and the other above) yields a triangle in which some of the given bisector lines are external, not internal bisectors).

Yet, for the given position of Aux1 and Aux2, we build the Incenter I, circumcenter O and orthocenter H of the triangle and then ask (through LocusEquation) for the position of vertex A such that the three points O, H, I are aligned. See Figure 4.

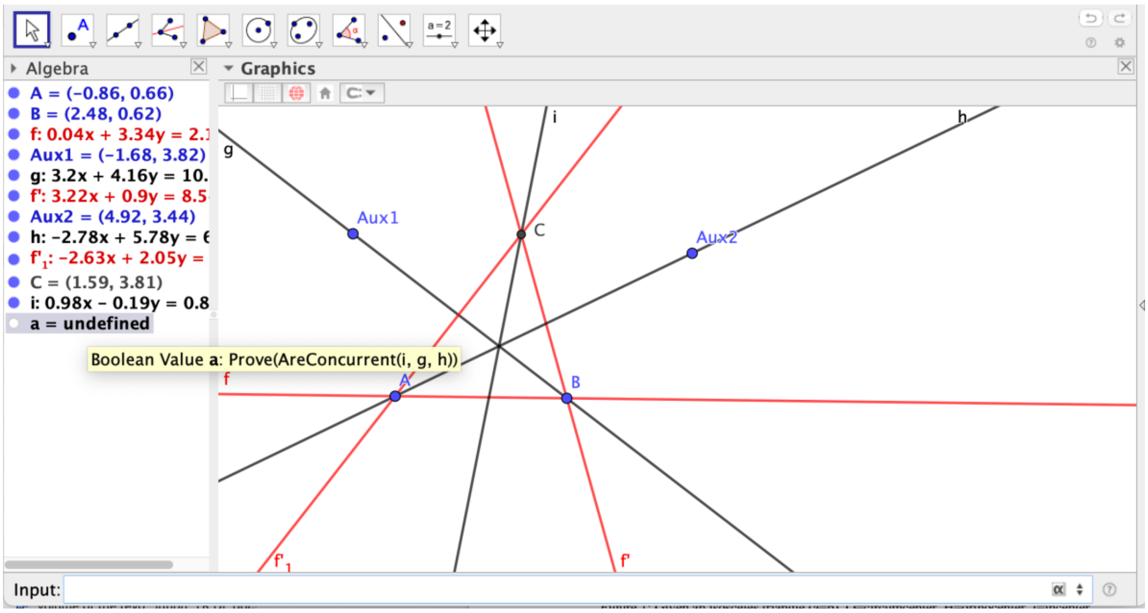

Figure 3: construction of a triangle given two bisector lines.

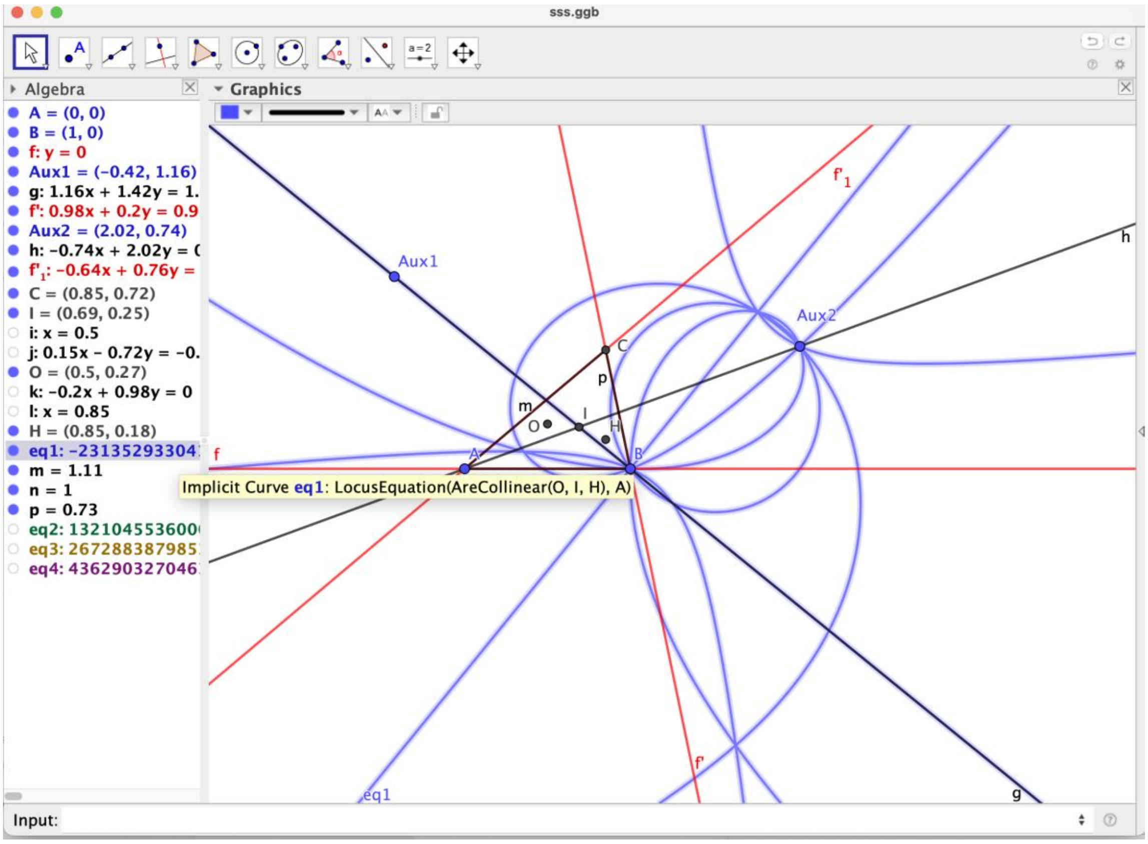

Figure 4: Locus of A so that O, I, H are aligned.

The output is a curve of degree 12, represented in blue in Figure 4. But what does this curve mean with respect to the triangle ABC? To answer this question we consider the segments m (side AC), n (side AB, that we have initially fixed as A(0,0), B(1,0), to help GeoGebra performing some calculations), p (side BC). And then we ask for the Locus of A so that m=p (see Figure 5).

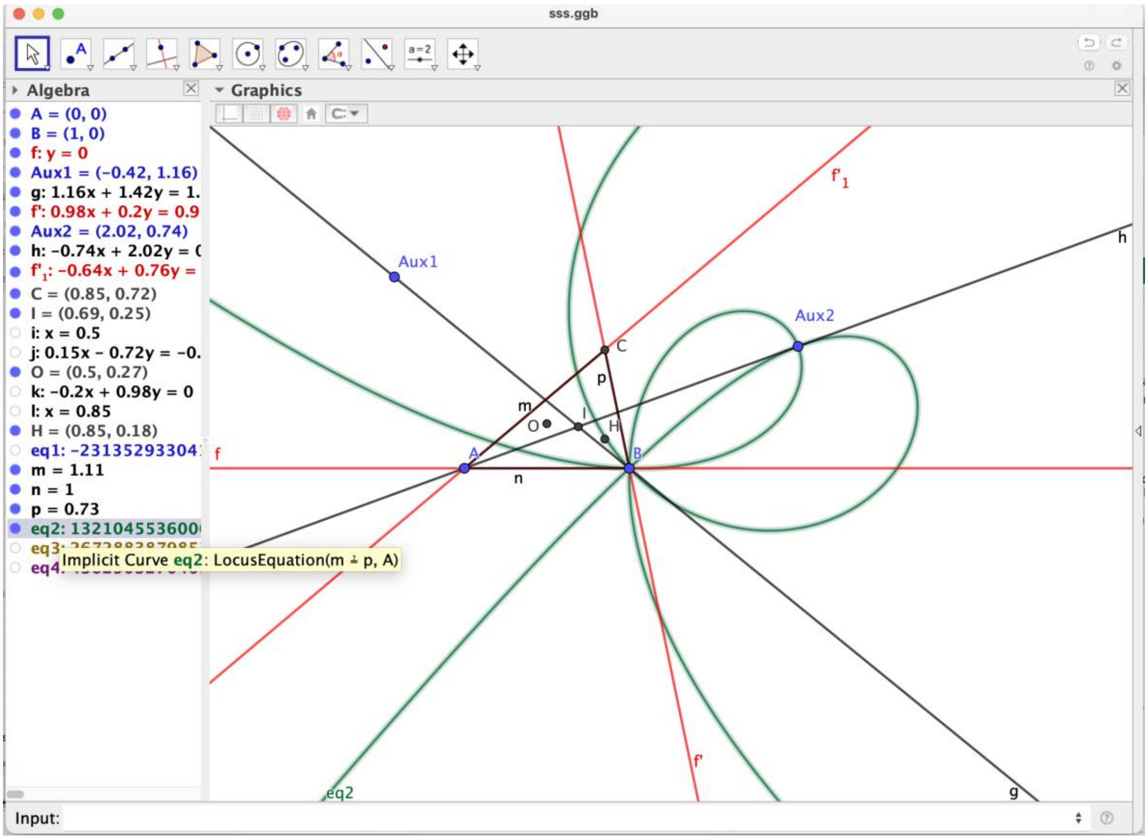

Figure 5: Locus of A so that AC=BC

Similarly, we construct the locus of A so that m=n (see Figure 6), and the locus of A so that n=p (AB=BC), see Figure 7. That is, the last three equations cover all possibilities for ABC being isosceles. Now we verify, visually and symbolically (by factoring eq1 and checking that each factor is indeed a factor of eq2, eq3 or eq4), see Figure 8, that the locus of A for the alignment of O, I, H, is contained in the locus for the triangle to be isosceles.

We observe that the equations eq2, eq3, eq4 strictly include the locus of the alignment (so that it seems that the converse: isosceles implies alignment, does not hold, but it is clear a question of eq2, eq3, eq4 including some degenerate cases or cases where the given incenters are actually excenters).

See the Maple worksheet or its PDF (*alignment O I H.pdf* and *alignment O I H.mw*), where the equations of the curves and their factors (over the rationals) are displayed.

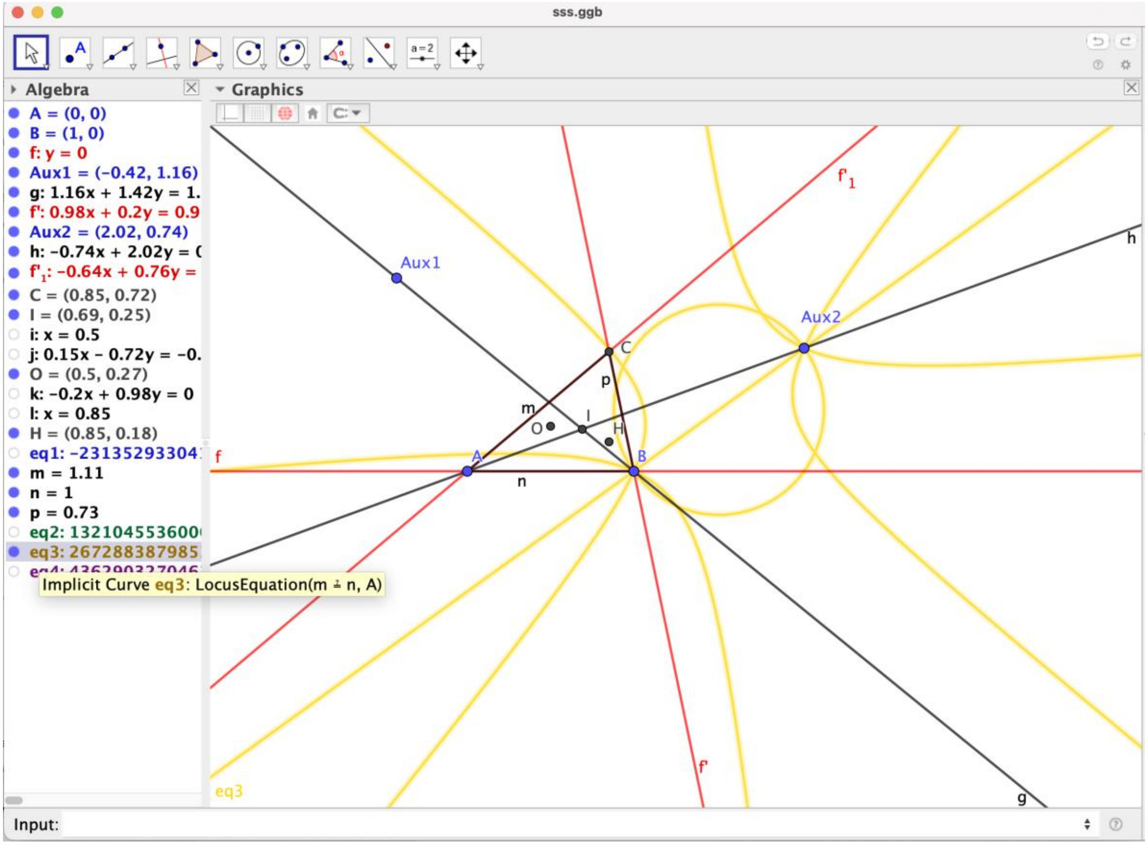

Figure 6: Locus of A so that AC=AB

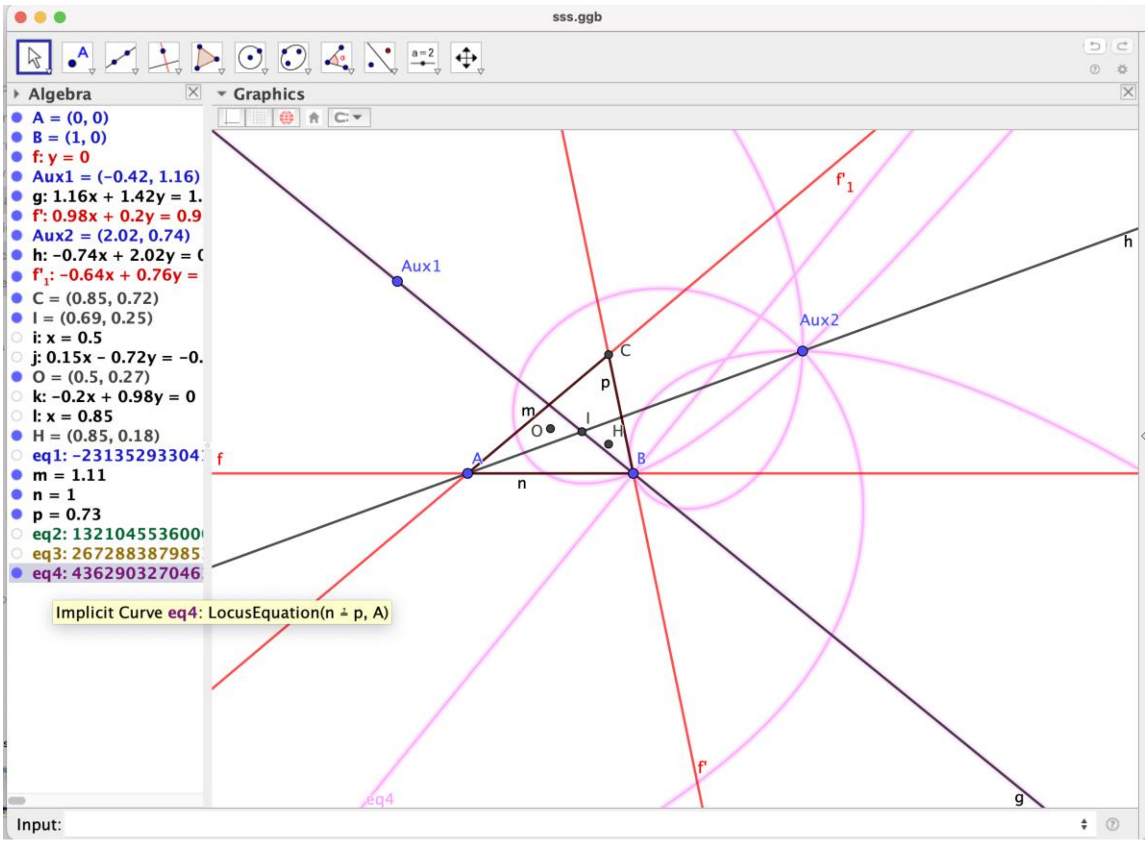

Figure 7: Locus of A so that AB=BC

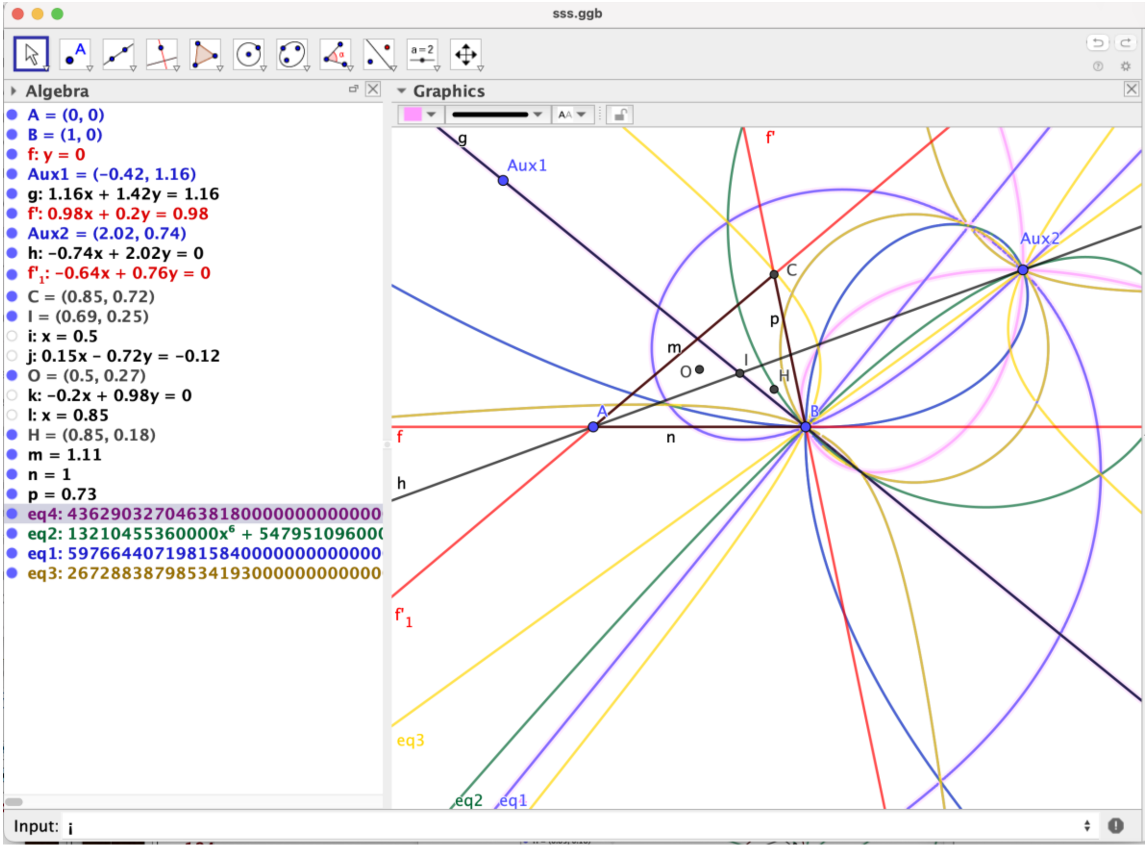

Figure 8: the four Loci for A (eq1=O I H aligned, eq2, eq3, eq4 = ABC isosceles)

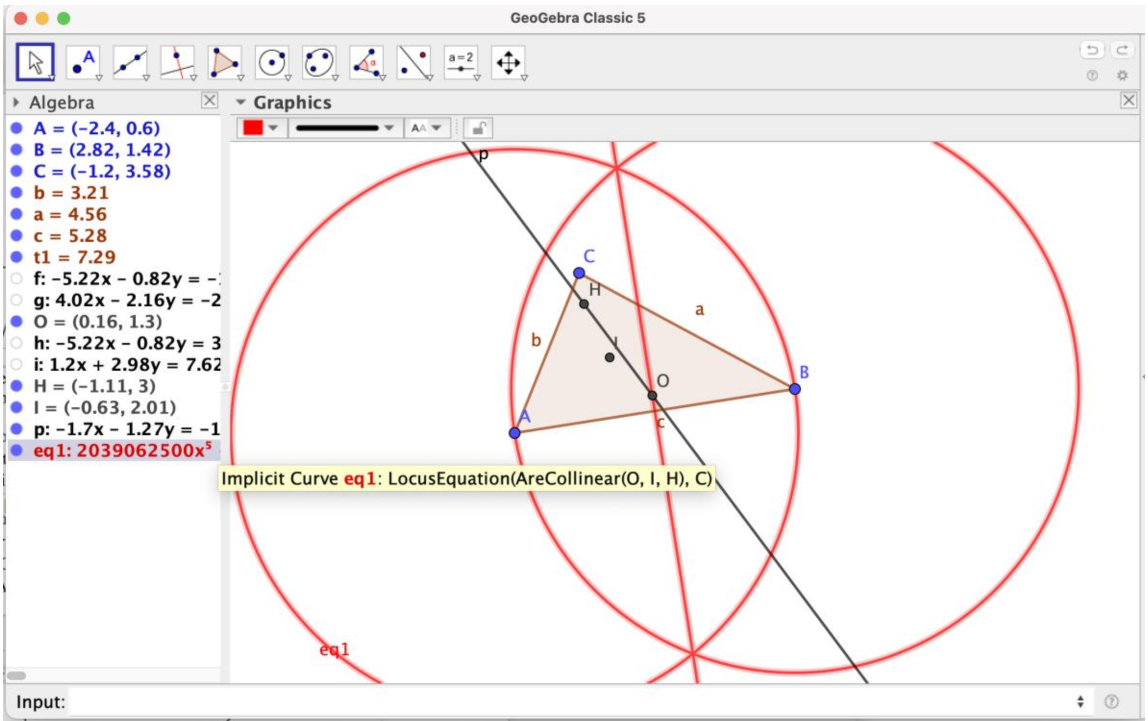

Figure 9: Using *IncenterCircle* to define I. Then, obtaining three Loci for vertex C when eq1=O I H aligned, yielding that ABC must be isosceles

On the other hand, using the *IncenterCircle* command to define the incenter, we obtain (see Figure 9) that the locus of vertex C for the collinearity of O, I, H, is the union of two circles (AC=AB and BC=BA) and a line (AC=BC), so the triangle must be isosceles. As mentioned before, using *IncenterCircle*, does not solve (yet) all the problems we have addressed in this context (yet!).

Of course, "isosceles implies alignment" is obvious, does not require proof, since in the construction we will use the same line for defining a height, a median, an angle bisector.

PROBLEM 2

Prove that the equilateral triangles have the smallest perimeter possible containing a given area.

This is a classical problem, quite easy to solve with the traditional means, but not so simple to be immediately solved with GeoGebra. Indeed, there is no way to display with GeoGebra the set of all triangles with a given (symbolic) area, unless one fixes at least one side (say, AB, as in Figure 10). Then, minimizing the perimeter is not a task that GeoGebra can accomplish automatically; it requires –for the moment– some "human intelligence".

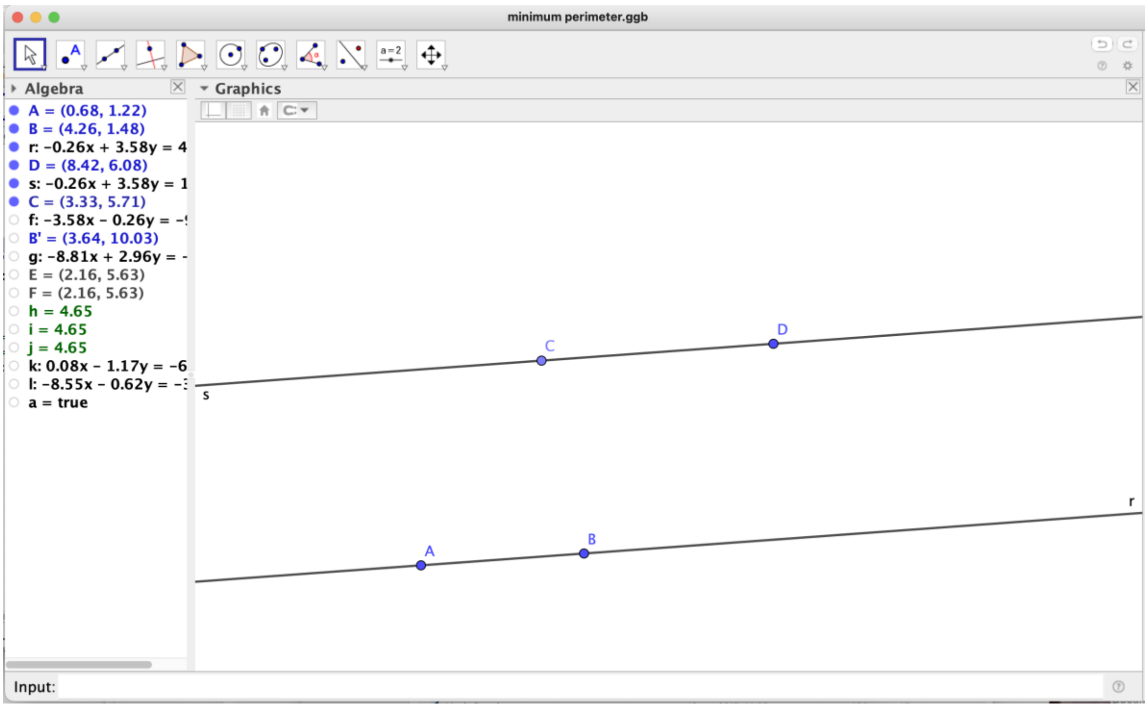

Figure 10: Side AB determining line r. Free point D and parallel line s to r through D. Point C in s.

PROBLEM: Given side AB and a parallel line to AB at a given distance, find (using GeoGebra) where to place vertex C in this line (so that the area of ΔABC will be equal for all positions of C) in such a way that the perimeter of ΔABC is minimum. See Figure 10.

HINT: Guess (as human) where to place C and have GeoGebra discovering that this position is the sought one.

SOLUTION TO PROBLEM 2

2.1 Classic solution

Let A, B be two fixed points on a line r. Let s be a line parallel to r at a fixed distance d. Let C be a point moving along line s. Let us show that the point C such that the perimeter of ΔABC is minimum is the intersection of s with the perpendicular bisector line of AB. The perimeter of ΔABC is AB + BC + CA, where AB is constant. Hence, we are being asked for BC + CA minimum. Let B′ be the reflection of B with respect to line s. It holds, BC + CA = B′C + CA Since A and B′ are fixed, we are being asked to find the point C on s such that B′C +CA is minimum. However, the shortest distance between two points is the one of the segments joining both points, so C = E = AB′ ∩ s, which is –by symmetry– precisely the intersection of s with the perpendicular bisector f of AB. See Figure 11.

Finally, to show that equilateral triangles have the smallest perimeter for a given area, assume that a triangle with the minimum perimeter is not equilateral, so there are at least two sides of different lengths. Then take the third side as side A, B in the previous paragraph and follow the argument there, concluding that the other two sides must have equal lengths, contradicting our assumption.

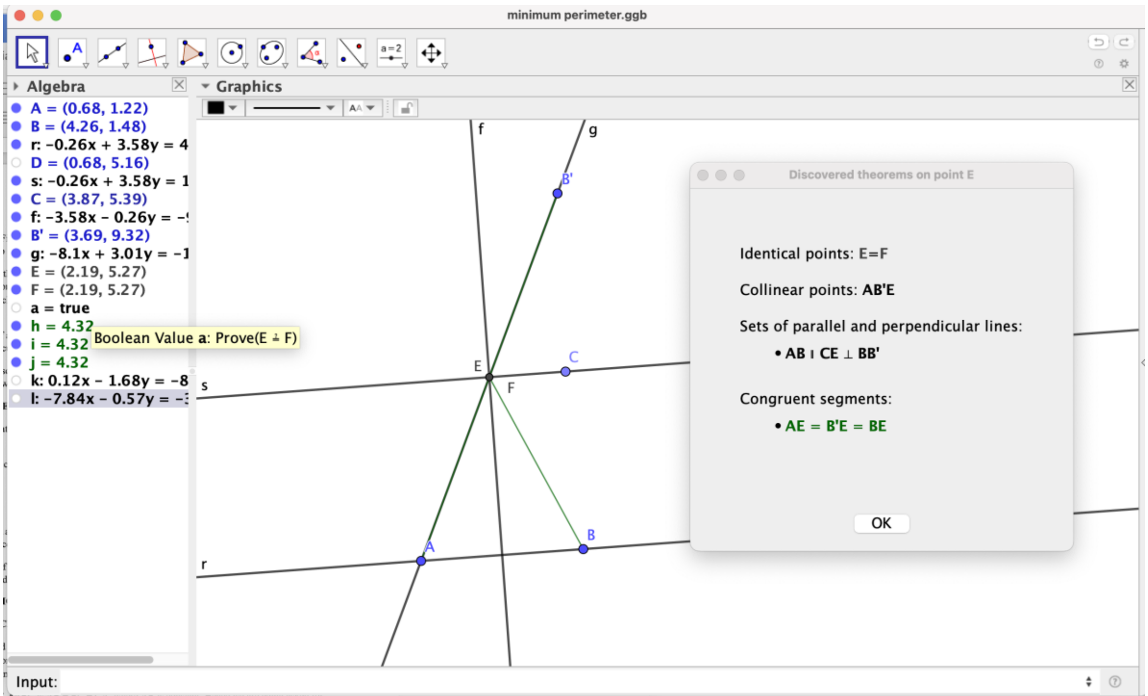

Figure 11: Automatic discovery of the coincidence of E (intersection of AB’ with s) and F (intersection of the bisector f with s).

2.2 Solving with GeoGebra

We start with two points A, B and the line r they define. Then choose a free point D and the parallel line s to r from D. Consider a point C on s. Then we try to find the position of C so that the perimeter AB+AC+BC is minimum. We follow the steps in the classical proof, building point B' symmetrical of B with respect to s and line g=AB'. Let f be the perpendicular bisector of AB and F the intersection of f and s, so that F is the position for C such that the triangle ABC is isosceles. Let be E the intersection of g and f, the position of the vertex C that minimizes the perimeter of ABC. Then ask GeoGebra Discover command to find properties involving point E. The answer includes, among others, the fact that E and F are identical!

Let us finish recalling the human readers of the eJMT 2022's, Kaput's visionary words [1]: technology should help us towards "a continuing transition from Doing (old) Things Better to Doing Better Things".

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