PROBLEM CORNER

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Problem 1

The Fibonacci sequence

$$F_n = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

has fascinated hobbyists, scientists, professional mathematicians and many others. One of the many properties of these numbers is $F_n^2 + F_{n+1}^2 = F_{2(n+1)}$. Call this property **P**. The generating function for the Fibonacci numbers is

$$\frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_n x^n.$$

Lets experiment with *Mathematica*. Change the generating function to

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Mathematica produces the sequence

$$a_n = 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, \ldots$$

Examine the sequence to see that $a_n^2 + a_{n+1}^2 = a_{2(n+1)}$. The sequence also has property **P**. In fact, showing that the sequence has property **P** was Putnam problem A3, 1999. If you look up the sequence in the *Online Encyclopedia of Integer Sequences*, OEIS, you see that this is sequence A000129 the sequence of Pell numbers.
For this first problem experiment with *Mathematica* to determine if the sequences corresponding to the following generating functions

$$\frac{1}{1-2^2x-x^2} = \sum_{n=0}^{\infty} a_n x^n \qquad \frac{1}{1-2^3x-x^2} = \sum_{n=0}^{\infty} a_n x^n$$

$$\frac{1}{1-3x-x^2} = \sum_{n=0}^{\infty} a_n x^n \qquad \frac{1}{1-3^2x-x^2} = \sum_{n=0}^{\infty} a_n x^n$$

have property **P** and if the sequence appears in OEIS. Use *Mathematica* to find an explicit formula for a_n in the generating function $\frac{1}{1-3^2x-x^2} = \sum_{n=0}^{\infty} a_n x^n$.

Problem 2
The previous problem now provides motivation for your next challenge. Consider the generating function

$$\frac{1}{1-m^k x - x^2} = \sum_{n=0}^{\infty} a_n x^n$$

where m and k are integers with $m \geq 2$ and $k \geq 1$. Use technology such as *Mathematica* to show that

$$\begin{aligned} a_n^2 + a_{n+1}^2 &= a_{2(n+1)} \\ &= \frac{\left(m^k + \sqrt{m^{2k}+4}\right)^{2n+3} - \left(m^k - \sqrt{m^{2k}+4}\right)^{2n+3}}{2^{2n+3}\sqrt{m^{2k}+4}}. \end{aligned}$$

Solution to Problem 1: *Mathematica* file `Problem.m` shows that the sequence $\{a_n\}$ corresponding to each generating function has property **P**. The sequence for the first, second, third and fourth generating function is sequence A001076, A041025, A006190 and A099371 respectively in OEIS. For the generating function $\frac{1}{1-3^2x-x^2} = \sum_{n=0}^{\infty} a_n x^n$ *Mathematica* produces the following explicit formula

$$a_n = \frac{\left(85 - 9\sqrt{85}\right)\left(9 - \sqrt{85}\right)^n + \left(85 + 9\sqrt{85}\right)\left(9 + \sqrt{85}\right)^n}{85 \cdot 2^{n+1}}.$$

Solution to Problem 2: The following solution uses *Mathematica*. First, use *Mathematica* to show that

$$a_n = \frac{\left(m^k + \sqrt{m^{2k}+4}\right)^{n+1} - \left(m^k - \sqrt{m^{2k}+4}\right)^{n+1}}{2^{n+1}\sqrt{m^{2k}+4}}.$$

Direct calculation of $a_{2(n+1)}$ and $a_n^2 + a_{n+1}^2$ using the above formula with *Mathematica* leads to the desired equation.

Appendix: The solutions in .m format are listed in plain text below:

```
(* Solution to Problem 1 *)
(* has property P, sequence A000045 in OEIS *)
FSeries=Series[1/(1 - x - x^2), {x, 0, 20}];
Print["FSeries=",FSeries];
(* has property P, sequence A000129 in OEIS *)
A2Series=Series[1/(1 - 2*x - x^2), {x, 0, 20}];
Print["A2Series=",A2Series];
(* has property P, sequence A001076 in OEIS *)
A2SQSeries=Series[1/(1 - 2^2*x - x^2), {x, 0, 20}];
Print["A2SQSeries=",A2SQSeries];
(* has property P, sequence A041025 in OEIS *)
A2CUSeries=Series[1/(1 - 2^3*x - x^2), {x, 0, 20}];
```

```
Print["A2CUSeries=",A2CUSeries];
(* has property P, sequence A006190 in OEIS *)
A3Series=Series[1/(1 - 3*x - x^2), {x, 0, 20}];
Print["A3Series=",A3Series];
(* has property P, sequence A099371 in OEIS *)
A3SQSeries=Series[1/(1 - 3^2*x - x^2), {x, 0, 20}];
Print["A3SQSeries=",A3SQSeries];
an=Assuming[n>=2 && Element[n, Integers], \
FullSimplify[SeriesCoefficient[1/(1 - 3^2 x - x^2), {x, 0, n}]]];
Print["an=",an];
aList=List[];
For[ j = 0, j <= 15, j++,
an=((85 - 9*Sqrt[85])*(9 - Sqrt[85])^j + \
(85 + 9*Sqrt[85])*(9 + Sqrt[85])^j)/(85*2^(1+j));
AppendTo[aList,Round[N[an]]];
];
Print[aList];
(* Solution to Problem 2 *)
an=Assuming[n>=2 && Element[n, Integers] && m>=2 && Element[m, Integers] \
&& k>=1 && Element[m, Integers], \
FullSimplify[SeriesCoefficient[1/(1 - m^k x - x^2), {x, 0, n}]]];
Print["an=",an];
f[n_] := ((m^k + Sqrt[4 + m^(2*k)])^(1 + n)- \
(m^k - Sqrt[4 + m^(2*k)])^(1 + n))/(2^(1+n)*Sqrt[4 + m^(2*k)]);
lhs=(f[n])^2+(f[n+1])^2;
rhs=f[2(n+1)];
Print["lhs=",FullSimplify[lhs]];
Print["rhs=",FullSimplify[rhs]];
(* Numerical test for particular values of m, k, n *)
For[ m = 3, m <= 5, ++m,
For[ k = 2, k <= 4, ++k,
For[ n = 1, n <= 10, n++,
lhs=(f[n])^2+(f[n+1])^2;
rhs=f[2(n+1)];
(*Print["lhs=",N[lhs]," rhs=",N[rhs]];*)
If[ lhs != rhs,
Print["Not Equal!!!"];
Exit[];
];
];
];
];
Print["Equal!!!"];
```

References

[1] Chen, H., Excursions in Classical Analysis, Mathematical Association of America, Inc., 2010.