PROBLEM CORNER

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We solve two loci problems proposed by T.Recio and C. Ueno in the February 2025 Problem Corner issue.

Problem 1. Consider a triangle ABC. Find the geometric locus of points P such that ∠PBA = ∠PCB and study its properties.

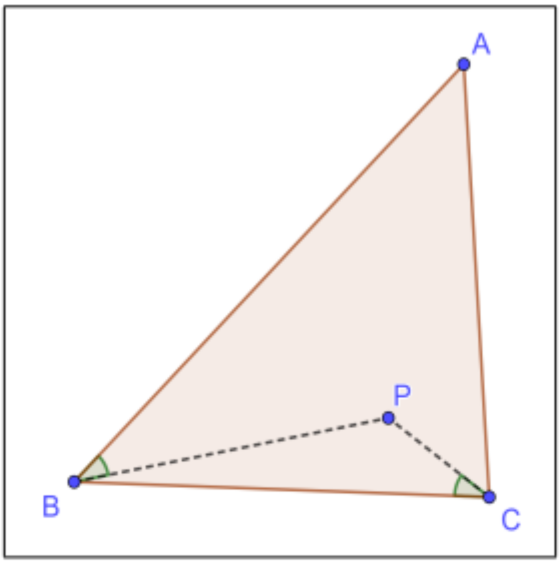

Figure 1: When $\angle PBA = \angle PCB$?

Solution.

Let $< BAC = \alpha$, $< ABC =$β, $< BCA = \gamma$, $< PCB = \theta$. Assume $\angle PBA = \angle PCB$. Then $< PBC = \beta - \theta$, and thus $< BPC = 180º - \theta - (\beta - \theta) = 180º - \beta$. Therefore, angle BPC is constant, and the locus of P is a circle (see figure).

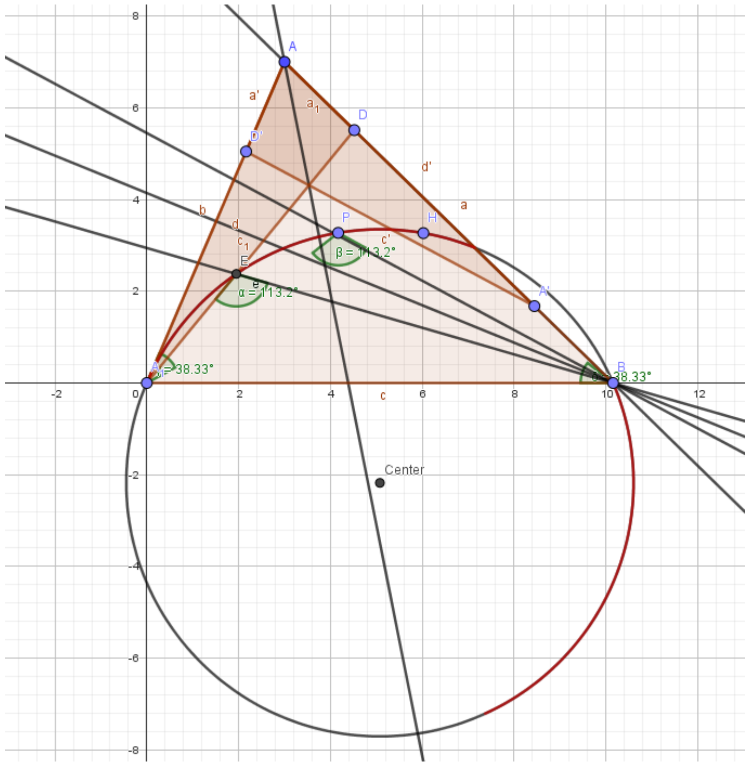

Figure 2: Locus of P for $\angle PBA = \angle PCB$

Problem 2 Consider a triangle ABC. Find the geometric locus of points P such that $\angle APB = \angle CPA$ and study its properties.

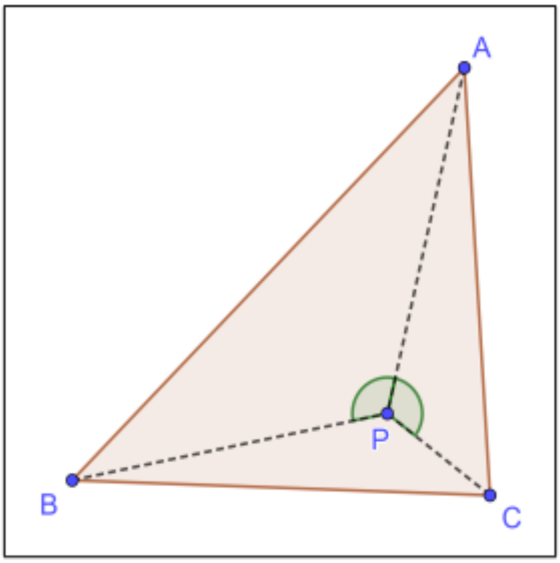

Figure 3: When does $\angle APB = \angle CPA$?

Let D be a point on BC, let AH be the altitude from A to BC and let $A'D'$ be the symmetric of AD with respect to the axis AH (see Figure 4, notice that $A=A'$).

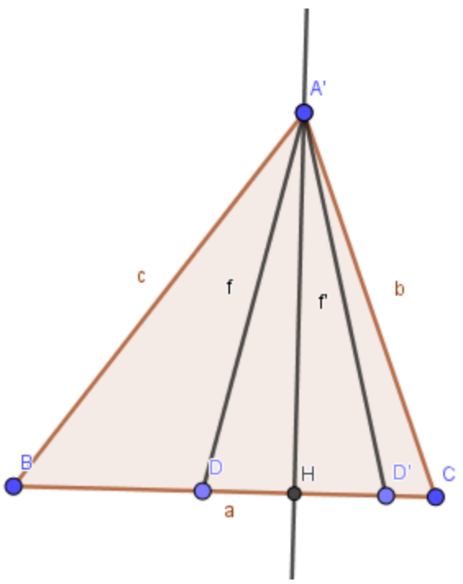

Figure 4

By construction we have $< BDA =< CD'A$, thus, the circles ADB and ADC intersect in both A and P and provide the solution to the problem, see Figure 5.

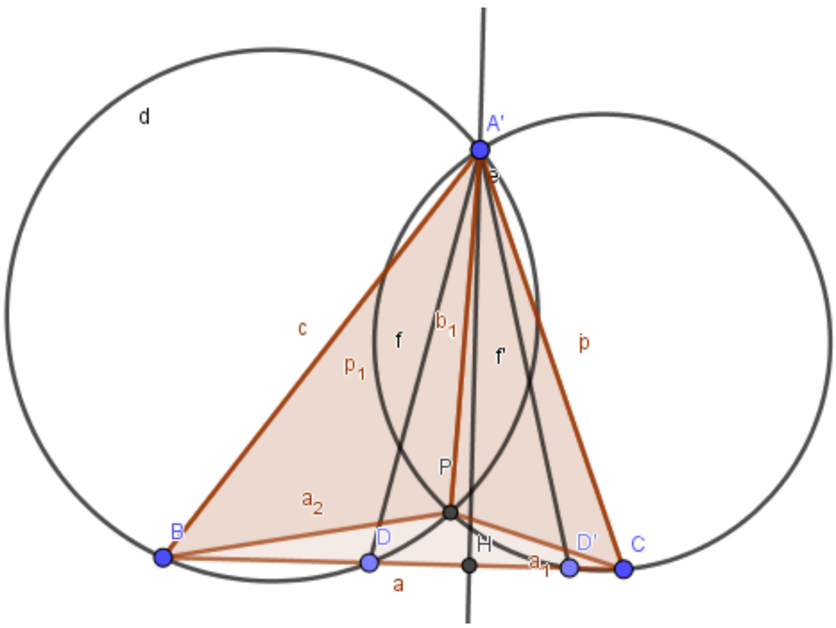

Figure 5: $< APB =< ADB =< A'D'C =< A'PC$

The locus of P is a curve going through points $C, H, A,$ that can be traced with GeoGebra (see Figure 6):

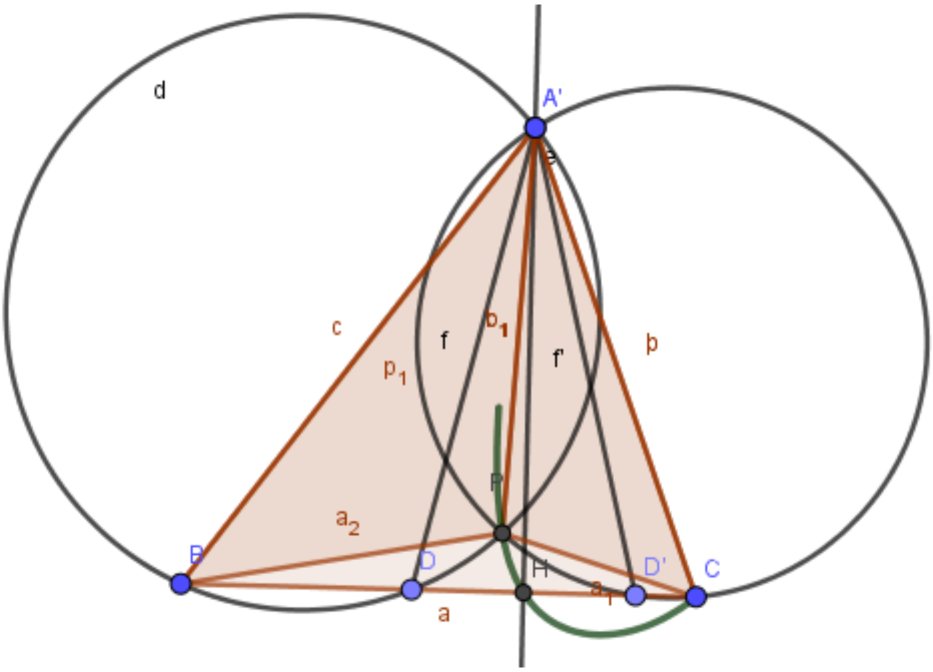

Figure 6: Tracing the locus of P when moving D

Finally, Figure 7 shows the equation of this locus, obtained with GeoGebra, a degree 5 polynomial that factors as a circle (degenerate case) and a cubic (an strophoid, see https://mathcurve.com/courbes2d.gb/strophoid/strophoid.shtml).

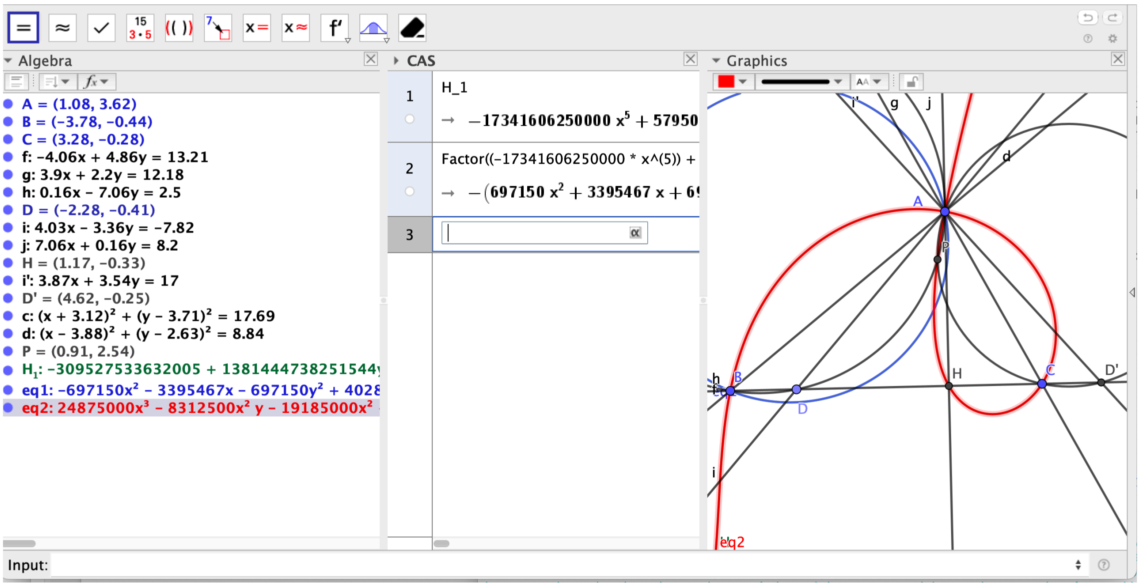

Figure 7: in red, the locus, an strophoid