

PROBLEM CORNER

Provided by Yuan YUAN

National Taichung University of Education, Taiwan

E-mail: yuanyuan@mail.ntcu.edu.tw

These two problems are suitable for exploration by mathematically gifted students ages 12-18.

Problem 1

The teacher wants to play a game with the students and will offer a reward (a bonus of NT\$1,000) or a punishment (singing a song) based on the result. The rules are as follows:

Each student is asked to choose a prime number between 300 and 400. Then, square that prime number and divide it by 6 to obtain the remainder (r).

Remainder (r)	Result
0	
1	
2	
3	
4	
5	

How should the teacher assign the reward or punishment in the table above so that every student's chosen prime number will result in singing a song? (Note: Choosing a prime between 300 and 400 is not essential.)

Solution:

Any integer p can be expressed in one of the forms $6n$, $6n+1$, $6n+2$, $6n+3$, $6n+4$, or $6n+5$, where n is an integer. All primes greater than 3 are of the form $6n+1$ or $6n+5$. Then:

$$(6n+1)^2 = 36n^2 + 12n + 1 \equiv 1 \pmod{6} \text{ and } (6n+5)^2 = 36n^2 + 60n + 25 \equiv 1 \pmod{6}.$$

Hence for every prime $p > 3$, $p^2 \equiv 1 \pmod{6}$. Therefore, if the teacher assigns 'sing a song' to the row of remainder ' $r=1$ ', all students will end up singing a song.

Note: For completeness, $2^2 \equiv 4 \pmod{6}$ and $3^2 \equiv 3 \pmod{6}$, noting that 2 and 3 are 'offending' primes falling out the designated range between 300 and 400.

Problem 2.

There are 25 prime numbers under 100. Use these prime numbers to create a number chain according to the following rules:

- (1) Use all 25 prime numbers **for chaining** (each **prime** can be used only once).
- (2) The leftmost digit of the next **prime** must match the rightmost digit of the previous **prime**. For example: 37–71–19–97...
- (3) The chain ends when no **further** primes can **follow** or when all primes have been used.

Questions:

- (1) What is the maximum number of primes that can be linked in the longest prime chain?
- (2) Which prime numbers can serve as the starting number of these longest chains?

Solution:

- The longest prime chain contains 13 prime numbers.
- The primes that can serve as the starting numbers of these longest chains are 2, 5, 41, and 61.

An example of one longest chain:

$$2 \rightarrow 23 \rightarrow 3 \rightarrow 31 \rightarrow 11 \rightarrow 13 \rightarrow 37 \rightarrow 7 \rightarrow 71 \rightarrow 17 \rightarrow 79 \rightarrow 97 \rightarrow 73$$

Discussion: Do you think that there is a similar pattern if we use a larger prime number?

This problem can be viewed as finding a Hamiltonian path in a directed graph where each vertex represents a prime number under 100, and an edge connects two vertices when the last digit of one prime equals to the first digit of the next.