Technology Makes Mathematics Learning More Challenging：Two Problems Responding Cycloid

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1, Introduction

The following two questions are from Problem Corner of Electronic Journal of Mathematics and Technology (eJMT), June, 2012:

Question 1： Are these two curves the same?

Given a circle O centered at point O with a radius 5 (we will name all the circles by the center points from now on), A is a fixed point on circle O and B is the point moving around on circle O. For each such point B, draw a circle centered at B with a radius |BA|. Curve 1 is the boundary of the green region generated by all the circles centered at B as shown in Figure 1

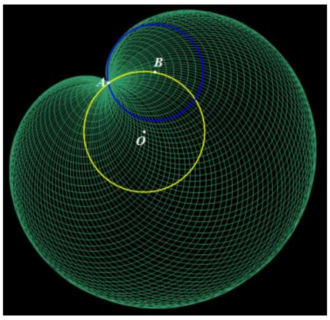

Figure 1

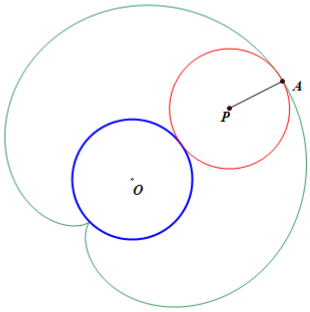

Figure 2

Circles O and P are both of radius 5 and tangent to each other at point A (the sharp edge). Being tangent to circle O, circle P rolls around circle O as illustrated in Figure 2. Curve 2 is the path of point A's movement.

Is curve 1 identical to curve 2? Prove your answer.

Question 2: Are these three curves the same?

Curve 3 is the edge of the purple region obtained by moving a line segment of length 5 along the x and y axes as shown in Figure 3.

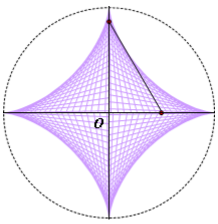

Figure 3

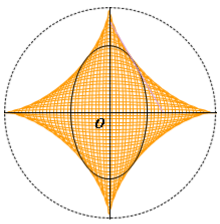

Figure 4

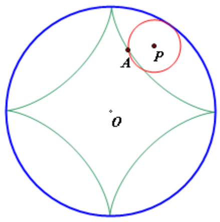

Figure 5

Curve 4 is the boundary of the orange region generated by all the ellipses E centered at the origin for which the sum of the lengths of the semi-major and semi-minor axes is always 5 as in Figure 4.

Given a circle O with a radius 5, P is another circle of radius 1.25 inside and tangent to circle O at point A (one of the sharp corners). Curve 5 is the trace of the movement of point A when P rolls around inside circle O as demonstrated in Figure 5.

Do curves 3, 4 and 5 represent the same curve? Prove your result.

From these two questions we can see that mathematics is becoming not only more interesting and applicable but also more challenging and theoretical. And they can help students realize the importance of reasoning and proof in mathematics.

In this paper we will solve these two fascinating problems.

2. The Envelope Curve of a Family of Curves

In geometry, an envelope curve of a family of curves in the plane is a curve that is tangent to each member of the family at some point. And for any point on the envelope curve, there is a common tangent line of both the envelope curve and one family curve (see [1]).

Let each curve C_t in the family be given by $f_t(x, y)=0$, where t is a parameter. Write $F(t, x, y)=f_t(x, y)$ and assume F is differentiable. The envelope curve of the family is then defined as the set of points for which

$$F(t, x, y)=\frac{\partial F}{\partial t}(t, x, y)=0$$

for some value of t, where $\partial F/\partial t$ is the partial derivative of F with respect to t.

(1) The Equation of Envelope Curve in Figure 1

Let C_t be the circle which passes A with center B, and assume the point A is (5,0) and that of point B is ($5\cos t$, $5\sin t$), then the equation of C_t is

$$[x-5\cos t]^2+[y-5\sin t]^2=[5-5\cos t]^2+[0-5\sin t]^2$$

Or equivalently,

$$x^2-10(x-5)\cos t+y^2-10y\sin t-25=0$$

Then the partial derivative of C_t is

$$(x-5)\sin t-y\cos t=0$$

So we can get the equation of envelope curve of family curve in Figure 1 as:

$$\begin{cases} x=10\cos t-5\cos(2t) \\ y=10\sin t-5\sin(2t) \end{cases} \qquad (1)$$

(2)The Equation of Envelope Curve in Figure 3

Let C_t be the line segment whose x interception is $5\cos t$ and y interception is $5\sin t$, then the equation of C_t is

$$\frac{x}{5\cos t}+\frac{y}{5\sin t}=1$$

or, equivalently,

$$x\sin t+y\cos t-2.5\sin(2t)=0$$

Then the partial derivative of C_t is

$$x\cos t-y\sin t-5\cos(2t)=0$$

So we can get the equation of envelope curve of family curve,

$$\begin{cases} x=5\cos^3 t \\ y=5\sin^3 t \end{cases} \tag{2}$$

(3)The Equation of Envelope Curve in Figure 4

Let C_t be the ellipse whose major semi-axis and minor semi-axis are t and 5-t respectively, where $0\leq t\leq 5$, then the equation of C_t is,

$$\frac{x^2}{t^2}+\frac{y^2}{(5-t)^2}=1$$

Then the partial derivative of C_t is

$$\frac{2x^2}{t^3}+\frac{2y^2}{(t-5)^3}=0$$

So we can get the equation of envelope curve of family curve,

$$\begin{cases} x^2=\dfrac{t^3}{5} \\ y^2=\dfrac{(5-t)^3}{5} \end{cases}$$

Or equivalently,

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=5^{\frac{2}{3}} \tag{3}$$

3. The Equation of Hypocycloid and Epicycloid

Hypocycloid (see [2]) is a plane curve illustrating the trace of a fixed point on a small circle that rolls within a larger circle. If the smaller circle has radius r, and the larger circle has radius R = kr, then the parametric equations for the curve can be given by:

$$\begin{cases} x(\theta)=(R-r)\cos\theta+r\cos(\dfrac{R-r}{r}\theta) \\ y(\theta)=(R-r)\sin\theta-r\sin(\dfrac{R-r}{r}\theta) \end{cases} \tag{a}$$

Epicycloid (see [3]) is a plane curve representing the path of a fixed point of a circle which slips around a bigger as in Figure 2. If the smaller circle has radius r, and the larger circle has radius R = kr, then the parametric equations for the curve can be given by:

$$\begin{cases} x(\theta) = (R+r)\cos\theta - r\cos(\dfrac{R+r}{r}\theta) \\ y(\theta) = (R+r)\sin\theta - r\sin(\dfrac{R+r}{r}\theta) \end{cases} \qquad \text{(b)}$$

So, from the equation (b) we can get the equation of cycloid curve in Figure 2 is

$$\begin{cases} x = 10\cos\theta - 5\cos(2\theta) \\ y = 10\sin\theta - 5\sin(2\theta) \end{cases} \qquad (4)$$

And from the equation (a) the equation of cycloid curve in Figure 5 is

$$\begin{cases} x = 3.75\cos\theta + 1.25\cos(3\theta) = 5\cos^3\theta \\ y = 3.75\sin\theta - 1.25\sin(3\theta) = 5\sin^3\theta \end{cases} \qquad (5)$$

4. Conclusion

(1) We can easily find that the formulae (1) and (4) are identical. So the edge of the traced regions in Figure 1 and the second curve in Figure 2 are the same.

(2) Equation (3) can be derived easily from expressions (2) and (5). So the edges of the two regions in Figure 3 and 4 are same as the curve in Figure 5.

5. Reference

[1] http://en.wikipedia.org/wiki/Envelope_(mathematics)
[2] http://en.wikipedia.org/wiki/Hypocycloid
[3] http://en.wikipedia.org/wiki/Epicycloid