PROBLEM CORNER

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Problem 1

A student uses a simple theodolite to measure the angle of elevation above the horizontal of a distant radio tower. The resulting angle is $\theta_1 = 1°25'29.907''$ above level. He then moves exactly $d = 50$ meters closer to the tower and measure the angle again, giving $\theta_2 = 1°25'55.546''$. What is the height of the tower h in meters, measured to the nearest centimeter? Be sure to include the curvature of the earth in your calculations, if necessary, with $R_e = 6371\ km$. Assume that the measurements are made from a height of $m = 2$ meters above the ground.

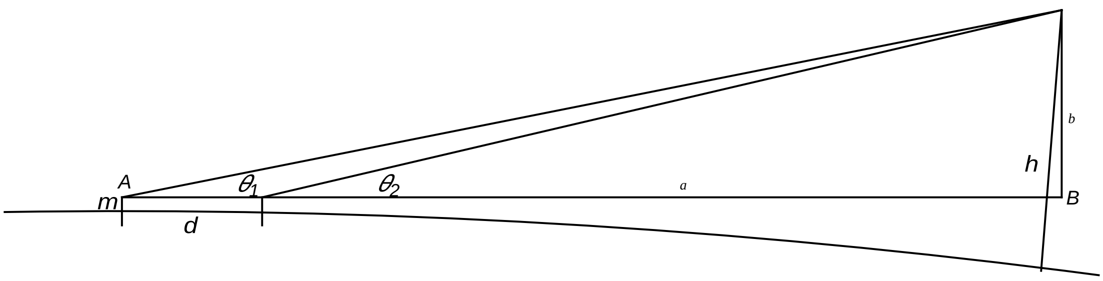

Figure 1 – Two triangles depicting the measurements

SOLUTION

We can first work with the right triangles, as shown in Figure 2, with base along the line tangent to the earth at the point of measurement. Basic trigonometry tells us that

$$\frac{b}{a} = \tan\theta_2 \qquad \text{and} \qquad \frac{b}{a+d} = \tan\theta_1$$

Since d is very small compared to Re, the effect of the Earth's curvature is not significant over this distance. Solving for b in the first equation and substituting into the second gives

$$b = a\tan\theta_2 \qquad \text{so} \qquad a\tan\theta_2 = (a+d)\tan\theta_1$$

Solving for a gives

$$a = d\frac{\tan\theta_1}{\tan\theta_2 - \tan\theta_1}$$

Computing with the given values shows that

$$a = 50\ \frac{\tan(1^{\circ}\ 25'29.907'')}{\tan(1^{\circ}\ 25'55.545'') - \tan(1^{\circ}\ 25'29.907'')} = 10{,}000.0\ meters$$

Then $b = 10{,}000\tan\left(1^{\circ}\ 25'55.545''\right) = 250\ meters$

As illustrated in Figure 2, the top of the tower is located $b + m = 252\ meters$ above the horizontal line representing the distance $A\,B = 10{,}050\ meters$ from the initial observation point. This distance is large enough that we need to take the curvature of the Earth into account in order to obtain an accurate measure of the height of the tower.

To determine *h* in the Figure 2, we construct a right triangle with base length equal to *A B* and height equal to $R_e + 252$.

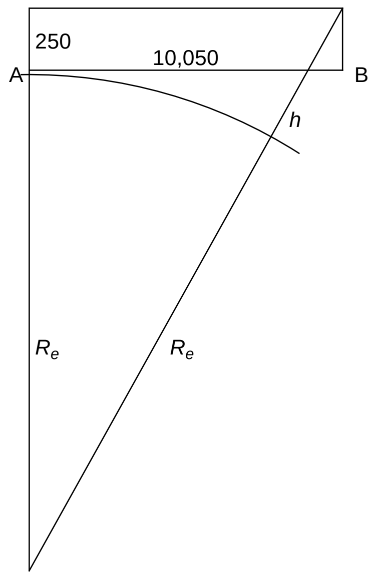

Figure 2 – A triangle with vertex at the center of the Earth (not to scale)

The hypotenuse of this triangle has length $R_e + h$. Using Pythagoras we obtain

$$h + R_e = \sqrt{(a+d)^2 + (R_e + m + b)^2}$$

Substituting our values and solving for *h* gives

$$h = \sqrt{10050^2 + 6371252^2} - 637100 = 259.962$$

So the height of the tower is 259.93 meters (to the nearest centimeter). Notice that $259.93 > 252.0$ so including consideration of the Earth's curvature was necessary to achieve the desired accuracy. In general, we can easily construct a function to solve this problem with any inputs of θ_1, θ_2, and *d* values. A file with the Mathematica code for such a function and a Word file for the same codes are available for download.

Problem 2

A trebuchet, as illustrated in Figure 3, is a medieval machine for launching stone projectiles against enemy castles. The path of such a projectile (as shown in Figure 4) is modeled by a parabolic path with equation: $y = a\,x^2 + b\,x + c$. A set of two dimensional (*x*, *y*) measurements are made and recorded in the table below, where, unfortunately, the *x* value for *y* = 631 is missing.

x	5	8	10	13	?
y	576	697	740	770	631

These points do not lie exactly on a parabola, so a linear least squares method was used to fit a parabola to the points by solving the matrix equation $A^T A\,\boldsymbol{x} = A^T \boldsymbol{b}$ where the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{b}$ represent the set of linear equations created by plugging the (*x*, *y*) coordinates of the points into the parabola equation $a\,x^2 + b\,x + c = y$.

The exact result of the least squares fit was the equation:

$$y = -\frac{7839578}{2592213}x^2 + \frac{135713223}{1728142}x + \frac{2692232981}{10368852}$$

A plot of the fitted equation along the four known points is shown in Figure 4. Your challenge is to determine exact value of the missing x coordinate of the 5th point such that the completed set of five points will give precisely this least squares solution.

Figure 3 - A Projectile Launcher (Trebuchet)

Image courtesy of www.medievalists.net

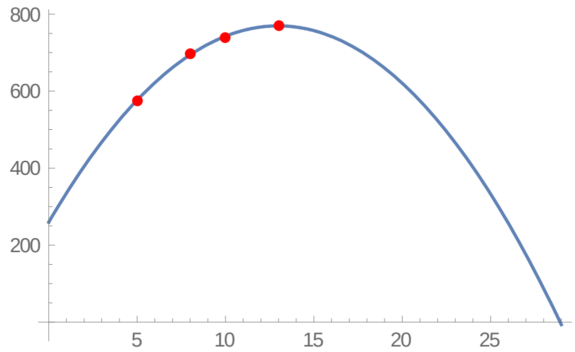

Figure 4 – The Projectile Path

Solution

We know that the equation, $y = -\frac{7839578}{2592213}\, x^2 + \frac{1357713223}{1728142}\, x + \frac{2692232981}{10368852}$ is a least squares solution to the system of linear equations given by substituting each of the points into the parabola equation $y = a\, x^2 + b\, x + c$, where x, y are the known point coordinates and a, b, c are the unknowns.

Making this substitution with our given points gives the system of linear equations

$$\begin{array}{ll} 25\, a + 5\, b + c = 576 & 169\, a + 13\, b + c = 770 \\ 64\, a + 8\, b + c = 697 & r^2 a + r\, b + c = 631 \\ 100\, a + 10\, b + c = 740 & \end{array}$$

To solve this we setup the linear system $A\, \boldsymbol{x} = \boldsymbol{b}$ with

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 100 & 10 & 1 \\ 169 & 13 & 1 \\ r^2 & r & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 576 \\ 697 \\ 740 \\ 770 \\ 631 \end{bmatrix}$$

We then apply the linear least squares technique by setting up the linear system $A^T A\, \boldsymbol{x} = A^T \boldsymbol{b}$ giving

$$\begin{bmatrix} r^4 + 43282 & r^3 + 3834 & r^2 + 358 \\ r^3 + 3834 & r^2 + 358 & r + 36 \\ r^2 + 358 & r + 36 & 5 \end{bmatrix} . \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 631\, r^2 + 263138 \\ 631\, r + 25866 \\ 3414 \end{bmatrix}$$

Substituting our known solution for a, b, c gives equality of the two vectors

$$\begin{bmatrix} -\frac{7839578}{2592213} r^4 + \frac{135713223}{1728142} r^3 + \frac{2692232981}{10368852} r^2 + \frac{454752654851}{1728142} \\ -\frac{7839578}{2592213} r^3 + \frac{135713223}{1728142} r^2 + \frac{2692232981}{10368852} r + \frac{22350385176}{864071} \\ -\frac{7839578}{2592213} r^2 + \frac{135713223}{1728142} r + \frac{31548945377}{10368852} \end{bmatrix} = \begin{bmatrix} 631\, r^2 + 263138 \\ 631\, r + 25866 \\ 3414 \end{bmatrix}$$

If r is a solution, it must satisfy the three polynomial equations enforced by the vector equality above. In particular, it must satisfy the third equality, which is the following quadratic equation.

$$-\frac{7839578}{2592213}\,r^2+\frac{135713223}{1728142}\,r+\frac{31548945377}{10368852}=3414$$

Clearing the fractions gives the quadratic equation

$$-31358312\,r^2+814279338\,r+1458536873=0$$

Completing the square or using the quadratic formula gives the two solutions

$$r=\frac{48738169}{7839578},\qquad r=\frac{79}{4}$$

While both solutions satisfy the first equality

$$-\frac{7839578}{2592213}\,r^4+\frac{135713223}{1728142}\,r^3+\frac{2692232981}{10368852}\,r^2+\frac{454752654851}{1728142}=631\,r^2+263138$$

only $r=\frac{79}{4}$, satisfies the all three equalities, including the second one

$$-\frac{7839578}{2592213}\,r^3+\frac{135713223}{1728142}\,r^2+\frac{2692232981}{10368852}\,r+\frac{22350385176}{864071}=631\,r+25866$$

To verify our solution we substitute $r=\frac{79}{4}$, into the equation $A\,\boldsymbol{x}=\boldsymbol{b}$, giving

$$\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 100 & 10 & 1\\ 169 & 13 & 1\\ \frac{6241}{16} & \frac{79}{4} & 1\end{bmatrix}\cdot\begin{bmatrix}a\\ b\\ c\end{bmatrix}=\begin{bmatrix}576\\ 697\\ 740\\ 770\\ 631\end{bmatrix}$$

We note that $a=-\frac{7839578}{2592213}, b=\frac{1357713223}{1728142}, c=\frac{2692232981}{10368852}$ is not a solution to the system above, in fact

$$\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 100 & 10 & 1\\ 169 & 13 & 1\\ \frac{6241}{16} & \frac{79}{4} & 1\end{bmatrix}\cdot\begin{bmatrix}a\\ b\\ c\end{bmatrix}\cong\begin{bmatrix}576.696\\ 694.343\\ 742.531\\ 769.45\\ 630.981\end{bmatrix}$$

so a, b, c is an approximate solution to the original system.

We now verify that a, b, c is an exact solution to the least squares system $A^T A\,\boldsymbol{x}=A^T\boldsymbol{b}$. Direct computation shows

$$A^T A\,\boldsymbol{x}=\begin{bmatrix}\frac{50030273}{256} & \frac{738415}{64} & \frac{11969}{16}\\ \frac{738415}{64} & \frac{11969}{16} & \frac{223}{4}\\ \frac{11969}{16} & \frac{223}{4} & 5\end{bmatrix}\cdot\begin{bmatrix}-\frac{7839578}{2592213}\\ \frac{1357713223}{1728142}\\ \frac{1357713223}{1728142}\end{bmatrix}=\begin{bmatrix}\frac{8148279}{16}\\ \frac{153313}{4}\\ 3414\end{bmatrix}=A^T\boldsymbol{b}$$

The Mathematica codes for any observations of (x, y) measurement can be found here, and a WORD file for the same codes can be found here.

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