The Problem Corner

José A. Vallejo
Email: `jvallejo@fc.uaslp.mx`

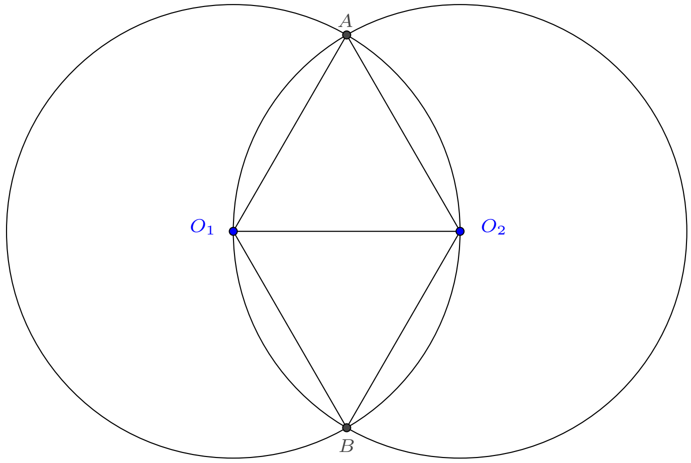
1. Modern cities are a mess. Every morning I have to decide what to do for going to work: To take the car or the subway. Of course, the car is worse (I am late one of every two times), but the subway is always crowded and uncomfortable (although I am late only one of every four times). I have not found a rule for predicting when I will be late due to the way of transportation, and for lack of a better algorithm what I do is to repeat the choice next day, when I am on time, and switch the transportation method next day when I am late. But I am not quite sure whether or not this is a good idea. For instance: What will be the probability of being late the day number 123?.

   Solution: If p_n is the probability of taking the car on the nth day, the probability of taking the subway that same day will be $1 - p_n$. But I will take the car either if I have taken it on the $(n-1)$th day and I have been on time (which happens one of every two times) or if I have taken the subway and have been late (one of every four times). Thus, we have the relation
$$p_n = \frac{1}{2}p_{n-1} + \frac{1}{4}p_{n-1} \, .$$
   This recurrence is very easy to solve 'by hand', but hey!, let's use Maxima (which includes the nice package `solve_rec` for this kind of problems):

   (%i1) load("solve_rec")$

   (%i2) solve_rec(p[n-1]/2+(1-p[n-1])/4-p[n],p[n]);
$$p_n = \frac{\%k_1}{4^n} - \frac{1}{34^n} + \frac{1}{3} \tag{\%o2}$$
   Thus, Maxima says that the probability of taking the car on the nth day is
$$p_n = \frac{1}{3}\left(1 - \frac{1}{4^{n-1}}\right) + \frac{1}{4^n}p_1 \, .$$
   For n large (such as 123), this probability approximates to
$$p_{123} \simeq \frac{1}{3} \, ,$$
   hence, the probability that I am late is
$$\frac{1}{2}\frac{1}{3} + \frac{1}{4}\frac{2}{3} = \frac{1}{3} \, .$$

2. In any divorce, there is a delicate matter: Who keeps the dog?. Olga and Oleg did not reach an agreement, so they take together the dog for a walk every evening. But they do not talk to each other. Each one put a different leash on the dog, one meter long, and they always keep a distance of one meter between them when walking. What is the surface in which the dog can move freely?.
   Solution: If O_1 is Olga's position, and O_2 is that of Oleg, supposing that the dog is leashed only to Olga, he could move inside a circle of radius π. The same would happen if he were leashed only to Oleg, see the figure (where the five segments all have length 1m).

   We want to compute the area of the intersection of the two circles. This is very difficult to do directly, but we can take the 'overlap and substract' approach: The area of the rhombus O_1AO_2B can be computed with elementary formulas (it is twice the area of one of the equilateral triangles with sides equal to 1), and is $\sqrt{3}/2$. The area of the circular sector O_1AB can be computed also easily: as the area of the whole circle is π, the portion comprised in the circular sector of central angle θ (in radians) is

$$A = \pi \frac{\theta}{2\pi} = \frac{\theta}{2} \,.$$

   In our case, as the central angle is determined by an equilateral triangle, $\theta = 2\pi/3$, so the area of the circular sector O_1AB is $\pi/3$. The same result is obtained for the area of the circular sector O_2AB. Thus, the area of the region given by the intersection of the two circles is (in m^2)

$$2\frac{\pi}{3} - \frac{\sqrt{3}}{2} \simeq 1.22837\,.$$

Figure 1 shows how to do this in GeoGebra. Notice the value of `Doggy'sArea`, computed as in the command line.

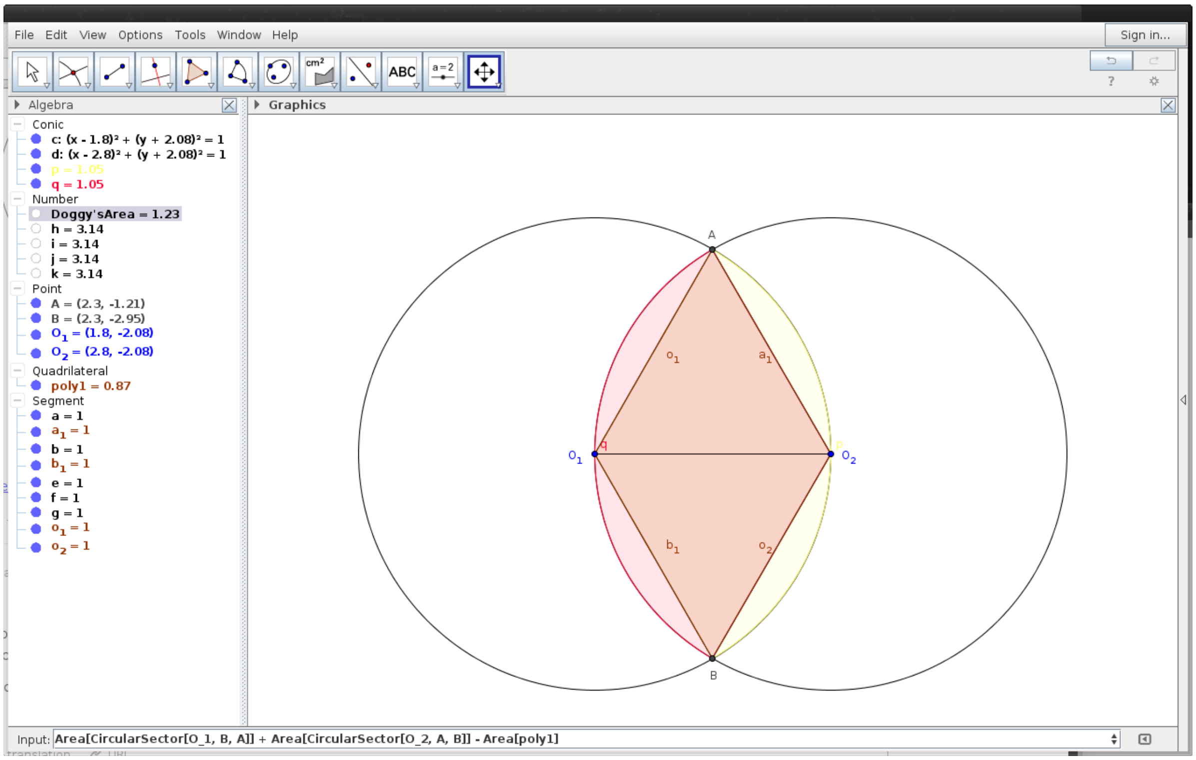


Figure 1: GeoGebra construction.