PROBLEM CORNER

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In view of problems posted at *[5]*, we post the corresponding problems in 3D.

Example 1 *We are given two concentric spheres centered at $O = (0, 0, 0)$ of radii of a and b (with $a < b$) respectively. See Figure 12, that is generated by GInMA [1] below. The unit sphere is depicted in blue and the sphere of radius 2 is the one in yellow. We are given a moving point A on the unit sphere and extend the ray OA to intersect the outer sphere at a point B. Next, we project point B onto the plane E (in purple), which is a plane that passes through A and is parallel to the xy plane. Denote by P the projection of point B in E. (In other words, the vector AP is perpendicular to the normal vector of the plane E.) Find the locus for the point P. Find the point B that will yield the maximum area for the triangle APB.*

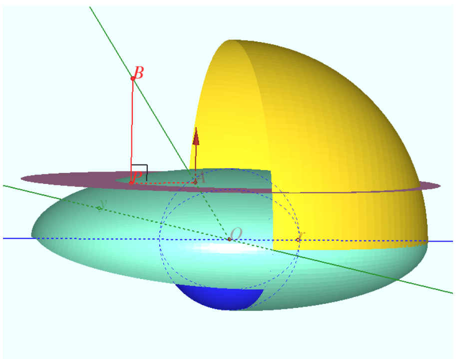

Figure 1. Generating an ellipsoid from two concentric spheres.

We write $A = (A_x, A_y, A_z), B = (B_x, B_y, B_z)$, and let $P = (P_x, P_y, P_z)$ be the locus point. We introduce the spherical coordinate system by letting φ the angle between OB and the positive $z-axis$, and the angle θ to be the angle between the projection of OB onto the xy-plane and the positive $x - axis$. If we let $a = OA$ and $b = OB$, then we see that $B_z = b\cos\varphi, B_x = b\sin\varphi\cos\theta$ and $B_y = b\sin\varphi\sin\theta$. We note that $P_z = A_z = a\cos\varphi$, $P_x = B_x = b\sin\varphi\cos\theta$ and $P_y = B_y = b\sin\varphi\sin\theta$. It shows that the locus surface in this case is an ellipsoid of the form

$$\frac{P_x^2}{b^2} + \frac{P_y^2}{b^2} + \frac{P_z^2}{a^2} = 1.$$

We may interpret part (a), finding the locus surface, as one way of constructing an ellipsoid as stated in *[3]*. We construct the locus surface in green as seen in Figure 1 with the help of *[1]*.

In order to generalize the idea of obtaining a locus through perpendicular projections, we replace the outer ellipsoid in Exercise 9 with another surface that encloses a sphere of a given radius. We consider the following cardioid surface below:

Example 2 *We are given a sphere centered at* $O = (0, 0)$ *with radius of* r_0*, and the cardioid surface* S*, by rotating* $[x(t), y(t)] = [a(1 - \cos t)\cos t + a, a(1 - \cos t)\sin t]$*, where* $t \in [0, 2\pi]$*, around the* $x - axis$*. Let* A *be a moving point on the sphere and we extend the ray* OA *to intersect the outer cardioid surface at a point* B*. Next, we project point* B *onto the plane* E*, which is a plane that passes through* A *and is parallel to the* xy *plane. Denote by* P *the projection of point* B *in* E*. In other words, the vector* AP *is perpendicular to the normal vector of the plane* E*. Find the locus for the point* P*. (See Figures 2(a)-2(c))*

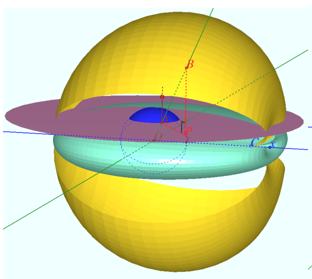

Figure 2(a) A sphere, cardioidal surface and locus

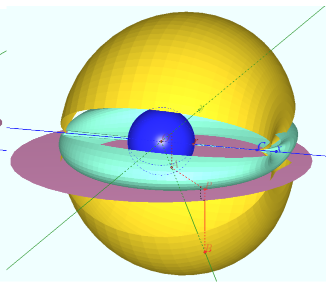

Figure 2(b) Locus surface when the point A varies

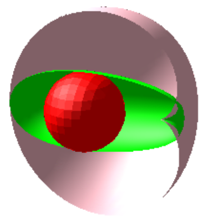

Figure 2(c) Locus generated by MAPLE

We note that the cardioid surface can be written as $[x(t), y(t)\cos\varphi, y(t)\sin\varphi]$,where $t \in [0, 2\pi]$ and $\varphi \in [0, \pi]$. As we have seen in *[5]* the locus for the two dimensional cardioid $[x(t), y(t)] = [a(1 - \cos t)\cos t + a, a(1 - \cos t)\sin t]$ is

$$[x^*(t), y^*(t)] = \left[a\left(\sin^2 t + \cos t\right), r_0\left(\frac{\sin t\,(1 - \cos t)}{\sqrt{2 - \cos^2 t}}\right)\right],$$

where $t \in [0, 2\pi]$. Thanks to symmetry, the locus surface for the cardioid surface is

$$[x^*(t), y^*(t)\cos\varphi, y^*(t)\sin\varphi].$$

In Figures 2(a) and 2(b), with the aid of GInMA *[1]*, we plotted various views of cardioid surfaces together with the enclosed spheres and respective locus surfaces. We also verified the locus surface analytically with [4] when $a = 2$ and $r_0 = 1$ as displayed in Figure 2(c).

References

[1] Geometry in Mathematical Arts (GInMA): *A Dynamic Geometry System*, see http://deoma-cmd.ru/en/Products/Geometry/GInMA.aspx.

[2] Geometry Expression, see http://www.geometryexpressions.com/.

[3] Sir William R. Hamilton, On the Construction of the Ellipsoid by two Sliding Spheres, Proceedings of the Royal Irish Academy, vol. 4 (1850), p. 341–342. https://www.emis.de/classics/Hamilton/Sliding.pdf.

[4] Maple: A product of Maplesoft, see http://maplesoft.com/.

[5] Problem Corner from the Electronic Journal of Mathematics and Technology, February 2019.

[6] Yang, W.-C. See Graphs. Find Equations. Myth or Reality? (pp. page 25-38). Proceedings of the 20th ATCM, the electronic copy can be found at this URL: http://atcm.mathandtech.org/EP2015/invited/2.pdf, ISBN:978-0-9821164-9-4 (hard copy), ISSN 1940-4204 (online version), Mathematics and Technology LLC.