PROBLEM CORNER

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Consider the unit circle and its inscribed regular hendecagon $A_1A_2A_3 \dots A_{10}A_{11}$.

Problem 1

Compute the product $|A_1A_2| \cdot |A_1A_3| \cdots |A_1A_{11}|$.

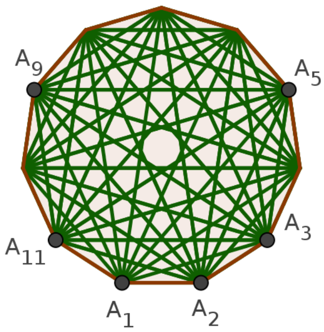

Figure 1 – a regular hendecagon and its diagonals

SOLUTION

Consider the complex plane. Without loss of generality we can assume that$A_1 = 1$. Translate the vertices of the hendecagon by one unit to the left. Now they are lying on a circle with center$-1$and radius 1, and the new position of $A_1$is $A_1{}' = 0$.So the translated vertices are the complex roots of the equation$(x+1)^{11} - 1 = 0$.After expanding the left hand side we learn that it is of form$x^{11} + \cdots + 11x$.By using Viète's formula according to the term$11x$it is clear that the sum of all products of the complex roots that contain exactly 10 factors, must be$\frac{11}{1} = 11$.Since all but one such products are 0 (because 0 is among the complex roots), we obtain that the product of the non-zero roots is 11. On the other hand, $|A_1A_2| \cdot |A_1A_3| \cdots |A_1A_{11}| = |A_1{}'A_2{}'| \cdot |A'_1A'_3| \cdots |A_1{}'A_{11}{}'| = |A_2{}'| \cdot |A_3{}'| \cdots |A_{11}{}'| = |11|$ that is 11.

Remark: The output of the GeoGebra CAS command

Numeric(Product((cos(2π / 11) + ί sin(2π / 11))^*k*-1,*k*,1,10)) allows us to make a conjecture that the product is 11. This computation cannot be achieved here symbolically, but we can still use another formulation. Namely,

Eliminate({Product(*z*^*k*-1,*k*,1,10)-*p*,*z*^11=1},{*z*}) delivers$\{-p^2 + 11p\}$that implies that the product is either 0 or 11.

Problem 2

Can you prove that the diagonals A_1A_5, A_2A_9 and A_3A_{11} are concurrent?

SOLUTION

It cannot be proven since this statement is false.

A closer look on the figure gives a visual evidence.

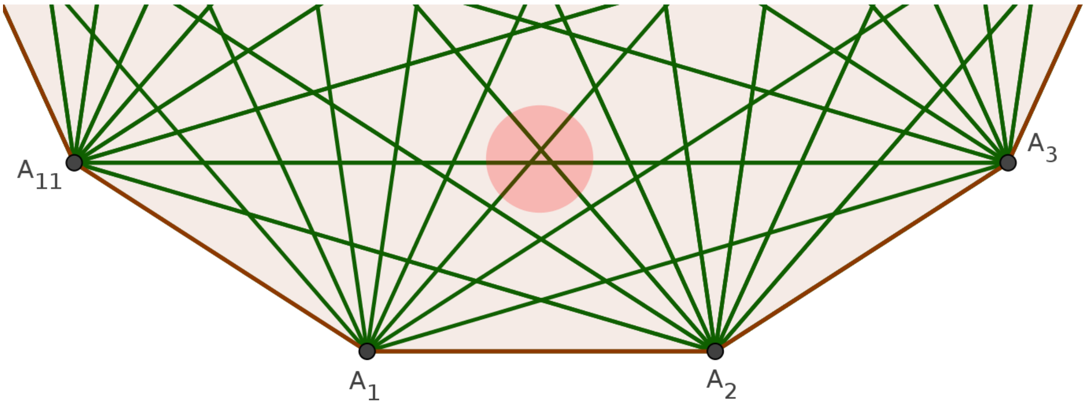

Figure 2 – the diagonals are not concurrent

We can also provide a symbolic proof that denies the statement.

Without loss of generality we can assume that $A_1 = (0,0)$, $A_2 = (1,0)$, and let us denote the coordinates of A_k by (x_k, y_k) for $k = 3, \ldots, 11$. In addition, denote the exact value of $cos\frac{2\pi}{11}$ by x and $sin\frac{2\pi}{11}$ by y. Now we can state that $\begin{pmatrix} x_k \\ y_k \end{pmatrix} - \begin{pmatrix} x_{k-1} \\ y_{k-1} \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \cdot \left(\begin{pmatrix} x_{k-1} \\ y_{k-1} \end{pmatrix} - \begin{pmatrix} x_{k-2} \\ y_{k-2} \end{pmatrix} \right)$, for $k = 3, \ldots, 11$. x and y can be expressed by the algebraic equations $32x^5 + 16x^4 - 32x^3 - 12x^2 + 6x + 1 = 0$ and $x^2 + y^2 = 1$. (See Watkins, W. and Zeitlin, J.: The minimal polynomial of cos(2π/n) in *The American Mathematical Monthly* 100(5):471–474 (1993) for more explanation on the quintic formula.) Also, if there is a point $P = (a, b)$ such that the points

A_1, A_5, P, A_2, A_9, P and A_3, A_{11}, P are collinear, respectively, then the following equations must also be true: $\begin{vmatrix} x_1 & y_1 & 1 \\ x_5 & y_5 & 1 \\ a & b & 1 \end{vmatrix} = 0$, $\begin{vmatrix} x_1 & y_1 & 1 \\ x_5 & y_5 & 1 \\ a & b & 1 \end{vmatrix} = 0$ and $\begin{vmatrix} x_3 & y_3 & 1 \\ x_{11} & y_{11} & 1 \\ a & b & 1 \end{vmatrix} = 0$.

Finally, the GeoGebra command **Eliminate({*x*_1 = 0, *y*_1 = 0, *x*_2 = 1, *y*_2 = 0, *x*_3 = – *x y*_2 + *x*_2 + *x x*_2 + *y y*_1 – *x x*_1, *y*_3 = *y*_2 + *x y*_2 + *y x*_2 – *x y*_1 – *y x*_1, *x*_4 = – *x y*_3 + *x*_3 + *x x*_3 + *y y*_2 – *x x*_2, *y*_4 = *y*_3 + *x y*_3 + *y x*_3 – *x y*_2 – *y x*_2, *x*_5 = – *x y*_4 + *x*_4 + *x x*_4 + *y y*_3 – *x x*_3, *y*_5 = *y*_4 + *x y*_4 + *y x*_4 – *x y*_3 – *y x*_3, *x*_6 = – *x y*_5 + *x*_5 + *x x*_5 + *y y*_4 – *x x*_4, *y*_6 = *y*_5 + *x y*_5 + *y x*_5 – *x y*_4 – *y x*_4, *x*_7 = – *x y*_6 + *x*_6 + *x x*_6 + *y y*_5 – *x x*_5, *y*_7 = *y*_6 + *x y*_6 + *y x*_6 – *x y*_5 – *y x*_5, *x*_8 = – *x y*_7 + *x*_7 + *x x*_7 + *y y*_6 – *x x*_6, *y*_8 = *y*_7 + *x y*_7 + *y x*_7 – *x y*_6 – *y x*_6, x_9 = – *x y*_8 + *x*_8 + *x x*_8 + *y y*_7 – *x x*_7, *y*_9 = *y*_8 + *x y*_8 + *y x*_8 – *x y*_7 – *y x*_7, *x*_{10} = – *x y*_9 + *x*_9 + *x x*_9 + *y y*_8 – *x x*_8, *y*_{10} = *y*_9 + *x y*_9 + *y x*_9 – *x y*_8 – *y x*_8, *x*_{11} = – *x y*_{10} + *x*_{10} + *x x*_{10} + *y y*_9 – *x x*_9, *y*_{11} = *y*_{10} + *x y*_{10} + *y x*_{10} – *x y*_9 – *y x*_9, 32*x*^5 + 16*x*^4 – 32*x*^3 – 12*x*^2 + 6*x* + 1=0, *x*^2 + *y*^2=1, Determinant({{*x*_1,*y*_1,1},{*x*_5,*y*_5,1},{*a*,*b*,1}}), Determinant({{*x*_2,*y*_2,1},{*x*_9,*y*_9,1},{*a*,*b*,1}}),**

Determinant({{*x*_3,*y*_3,1},{*x*_{11},*y*_{11},1},{*a*,*b*,1}})},{*a*,*b*}) gives {1} as output which means that the equation system is equivalent to the equation 0=1. This contradiction shows the falsity of the statement.

Alternatively, instead of **Eliminate**, the **Solve** command can also be used. In that case the empty output set implies the same conclusion.