PROBLEM CORNER

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Problem 1

MathCityMap (MCM, https://mathcitymap.eu/) allows to solve tasks which are bound to coordinates. The MCM app on today's smartphones uses a global navigation satellite system (GNSS) to determine the current position. GNSS uses the intersection of spheres in the 3D space. Determine the algebraic solution of the intersection of three spheres assuming that it exists. Feel free to use a Dynamic Mathematics Software (DMS, i.e., GeoGebra) to represent the problem visually.

SOLUTION

We will consider the following three spheres, where r_1, r_2, r_3, x_2 and y_3 can be freely chosen (slider in GeoGebra) while x_3 is dependent and will be determined later, as they can be easily constructed in a DMS and an affine change of basis can transform any three spheres into the chosen ones.

S_1 $$x^2 + y^2 + z^2 = r_1^2 \qquad (1)$$

S_2 $$(x - x_2)^2 + y^2 + z^2 = r_2^2 \qquad (2)$$

S_3 $$(x - x_3)^2 + (y - y_3)^2 + z^2 = r_3^2 \qquad (3)$$

Let us start with the spheres S_1 and S_2 by combining (1) and (2) using (2) – (1).

$$(x - x_2)^2 - x^2 = r_2^2 - r_1^2$$

$$\Leftrightarrow x^2 - 2x_2x + x_2^2 - x^2 = r_2^2 - r_1^2$$

$$\Leftrightarrow -2x_2x = r_2^2 - r_1^2 - x_2^2$$

$$\Leftrightarrow x = \frac{x_2^2 + r_1^2 - r_2^2}{2x_2} \qquad (4)$$

We define x_3 to this value

$$x_3 := \frac{x_2^2 + r_1^2 - r_2^2}{2x_2}$$

and conclude that the intersection of the spheres S_1 and S_2 lies in the plane P parallel to yOz through $(x_3, 0,0)$, i.e., having the equation $x = x_3$. Putting (4) into (1) gives

$$y^2 + z^2 = r_1^2 - x^2 = r_1^2 - \left(\frac{x_2^2 + r_1^2 - r_2^2}{2x_2}\right)^2 = \frac{4x_2^2r_1^2 - (x_2^2 + r_1^2 - r_2^2)^2}{4x_2^2}$$

which is a circle with centre $(0,0)$ and radius

$$r = \frac{1}{2x_2}\sqrt{4x_2^2r_1^2 - (x_2^2 + r_1^2 - r_2^2)^2}$$

i.e., having the equation

C_{12} $$y^2 + z^2 = r^2 \tag{5}$$

In the plane P the equation (3) simplifies to

C_3 $$(y - y_3)^2 + z^2 = r_3^2 \tag{6}$$

which is a circle with center $(y_3, 0)$ and radius r_3.

The intersection of the three spheres has thus been reduced to the intersection of two circles which is solved in a similar way by combining (5) and (6) using (6) – (5).

$$(y - y_3)^2 - y^2 = r_3^2 - r^2$$

$$\Leftrightarrow y^2 - 2y_3y + y_3^2 - y^2 = r_3^2 - r^2$$

$$\Leftrightarrow -2y_3y = r_3^2 - r^2 - y_3^2$$

$$\Leftrightarrow y = \frac{y_3^2 + r^2 - r_3^2}{2y_3} \tag{7}$$

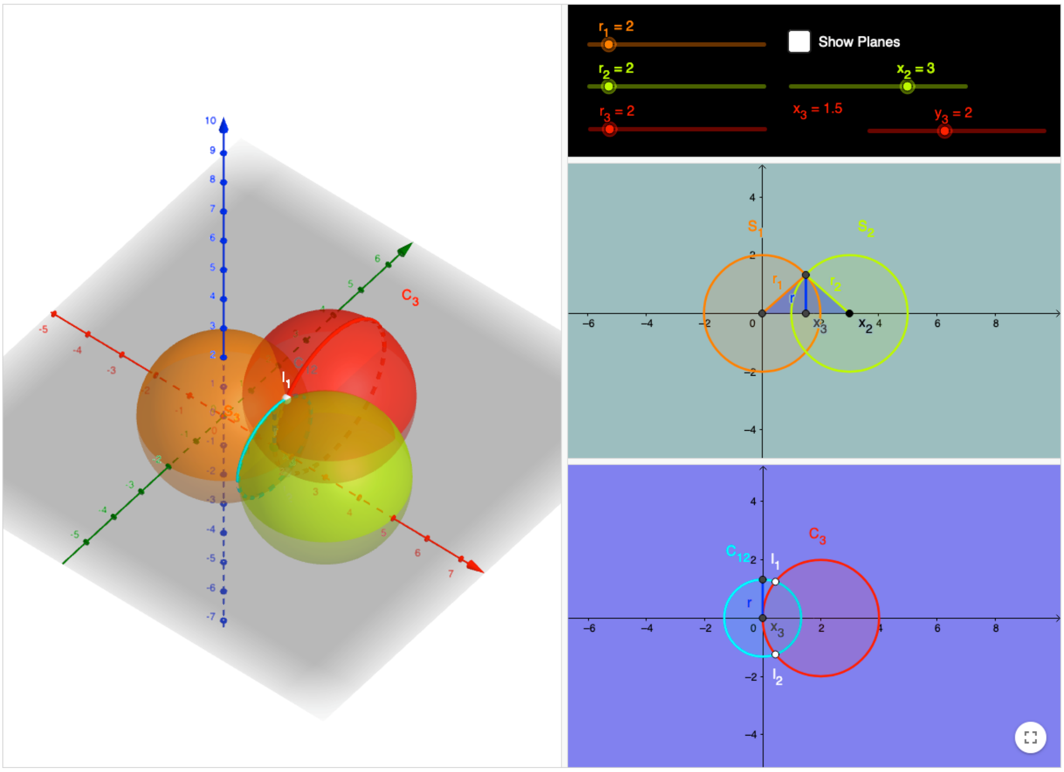
Putting (7) into (5) gives

$$z^2 = r^2 - y^2 = r^2 - \left(\frac{y_3^2 + r^2 - r_3^2}{2y_3}\right)^2 = \frac{4y_3^2r^2 - (y_3^2 + r^2 - r_3^2)^2}{4y_3^2}$$

which solves to

$$z = \pm\frac{1}{2y_3}\sqrt{4y_3^2r^2 - (y_3^2 + r^2 - r_3^2)^2}$$

An interactive GeoGebra Activity is available at
https://www.geogebra.org/m/hfujpcyw.



Regarding GNSS, one of the two solutions can be eliminated as it is geometrically incoherent. Nevertheless, a fourth satellite is necessary, as it provides solutions in the measurement of signal propagation time, due to desynchronization of the receivers' time compared to the satellites' atomic clocks.

Problem 2

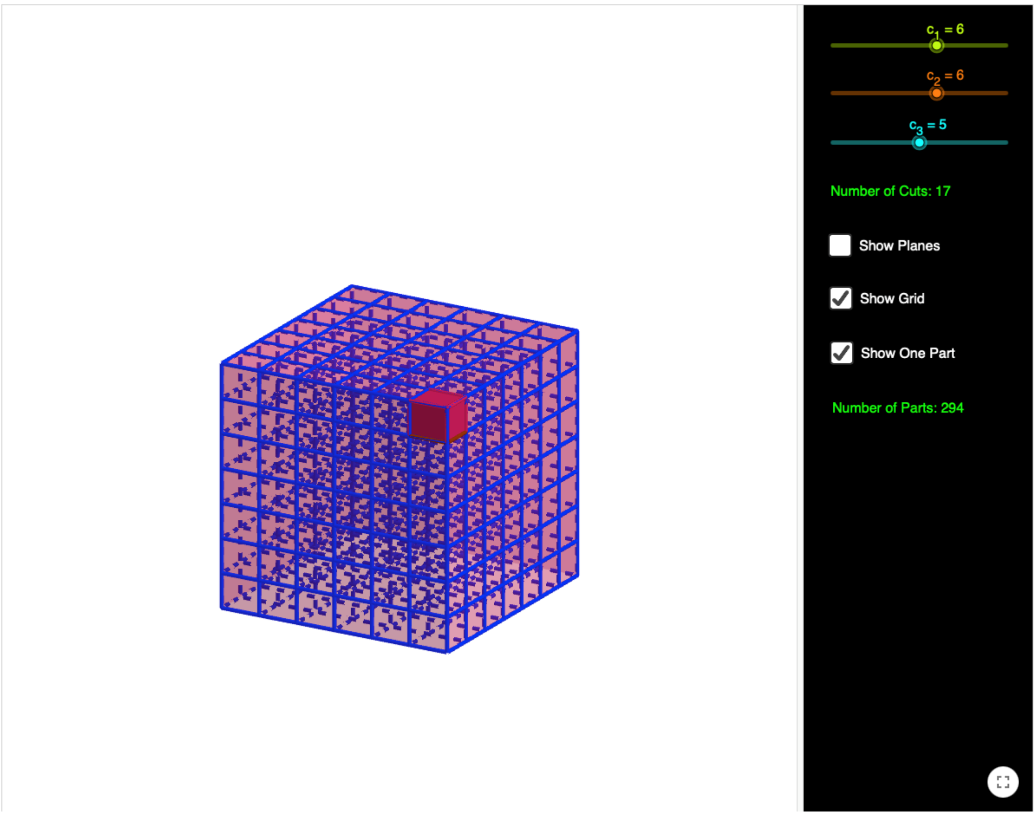
The preparation of a 3D print sometimes requires cutting a 3D object (multiple times) using a plane. Although the objective generally is not to obtain identical pieces, this is however interesting from a mathematical point of view! Find the maximum number of identical pieces obtainable by cutting a cube using 17 cutting planes. Feel free to use a Dynamic Mathematics Software (DMS, i.e., GeoGebra) to represent the problem visually.

Solution

We cut the cube with planes perpendicular to one side of the cube.

Let us first consider using all cuts on the same side of the cube: one cut will create two parts; three cuts will create four parts and in general c cuts will create $c + 1$ parts of the cube.

Let us then consider alternating the three spacial directions: the first cut will create two parts; the second cut will double to four parts; the third cut will double again to eight parts and in general if we make c_1, c_2, c_3 cuts in the respective directions then we will have $(c_1 + 1)(c_2 + 1)(c_3 + 1)$ parts of the cube.



We know that the total number of cuts is $c_1 + c_2 + c_3 = 17$ and we want to maximise the number of parts $(c_1 + 1)(c_2 + 1)(c_3 + 1)$. Thus, the solution is any permutation of the triple $(6,6,5)$, i.e., $(6,6,5), (6,5,6), (5,6,6)$, with a total number of parts $(6 + 1)(6 + 1)(5 + 1) = 7 \cdot 7 \cdot 6 = 294$.

An interactive GeoGebra Activity is available at https://www.geogebra.org/m/chw7t76s.