PROBLEM CORNER

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Problem 1

Little John suggests a new method on constructing a regular 13-gon by using a compass and a ruler (see Figure 1):

1. Draw a circle *c* of radius 100 mm.
2. Choose an arbitrary point *A* on circle *c*.
3. Draw a circle *d* of radius 187 mm with center *A*.
4. Mark the intersection points *B* and *M* of circles *c* and *d*.
5. Draw a circle *e* of radius 187 mm with center *B*.
6. Mark the other intersection point *C* of circles *c* and *e*.
7. Draw a circle *f* of radius 187 mm with center *C*.
8. Mark the other intersection point *D* of circles *c* and *f*.
9. And so on, mark further intersection points *E*, *F*, *G*, *H*, *I*, *J*, *K* and *L*.
10. Now *AIDLGBJEMHCKF* is a regular 13-gon.

We have the feeling that this cannot be accurate. Why? Explain the situation.

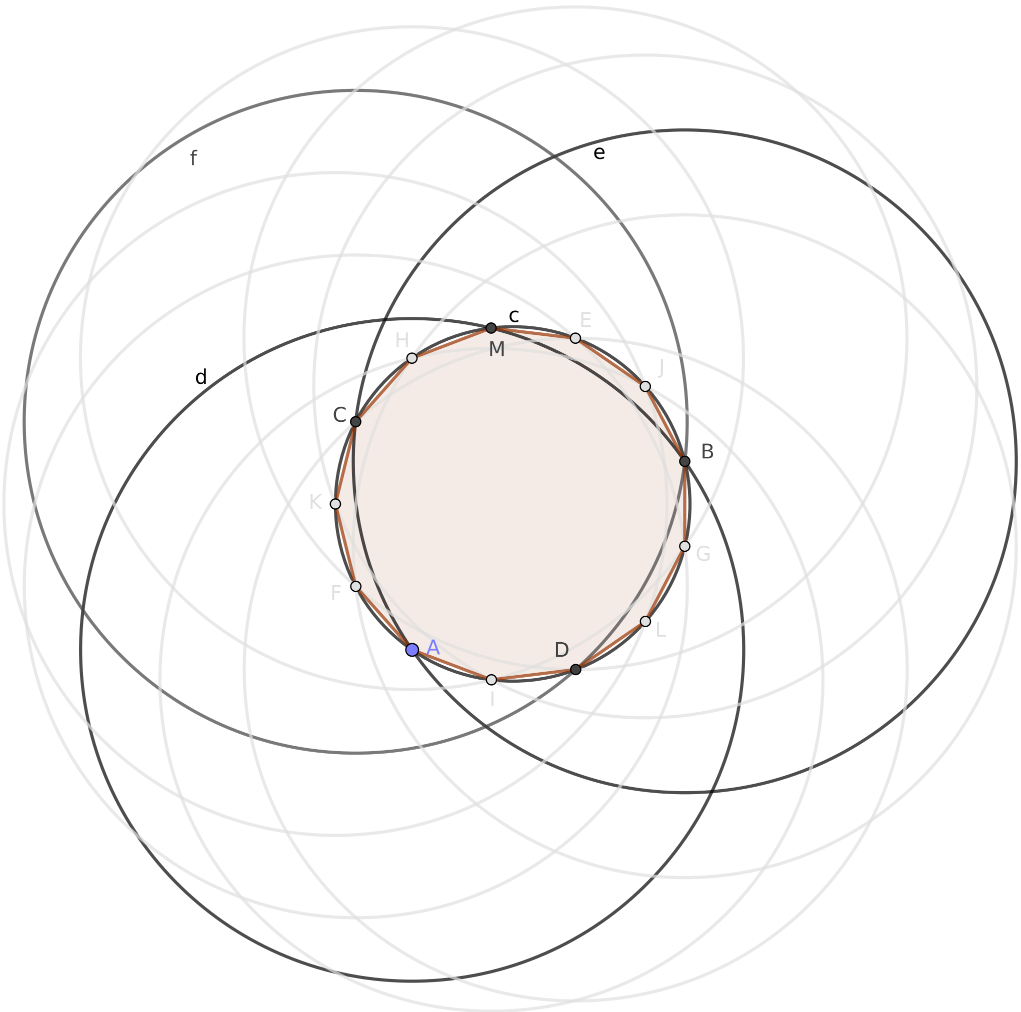

Figure 1 – Little John's method to construct a regular 13-gon

Solution

According to the Gauss-Wantzel theorem (see https://en.wikipedia.org/wiki/Constructible_polygon) it is impossible to construct a regular 13-gon by using compass and ruler. The number 13 is a prime but not of the form $2^{2^k}+1$ hence a regular 13-gon is non-constructible.

On the other hand, the ratio of the radius and the *s*-th diagonal of a regular *n*-gon can be computed with the formula $\sqrt{\left(1-\cos\left(\frac{2\cdot s\cdot\pi}{n}\right)\right)^2+\sin\left(\frac{2\cdot s\cdot\pi}{n}\right)^2}$. This computation can be derived by considering the unit circle and an inscribed regular *n*-gon $A_0A_1...A_{n-1}$, by letting $A_0 = (1,0)$, and projecting A_s on the *x*-axis to obtain point X_s, and finally considering the right triangle $A_0A_sX_s$ and using the Pythagorean theorem to express A_0A_s.

Putting s=5, n=13 in the formula we obtain 1.870032... By considering this number as a ratio, it is very close to 187/100. Hence, Little John's method finds every 5[th] diagonal in an almost-regular 13-gon. including side lengths of 47.91 and 47.79 mm, for 8 and 5 sides, respectively (see also a GeoGebra applet at https://www.geogebra.org/m/sdkhmwhp for further reference). Their difference is significantly less than 1 mm, or, in other words, it is below 0.25%.

Problem 2

Assume we would like to use Little John's method to construct exact regular n-gons by considering two numbers as input radii, r_1 and r_2 (in Problem 1, $r_1 = 100$, $r_2 = 187$, $n = 13$). Find all natural numbers n and all associated integer numbers r_1 and r_2 that indeed produce an exact regular n-gon with this method.

Solution

We say that two non-zero real numbers a and b are *commensurable* if their ratio a/b is a rational number.

We are going to use Vincenzi's theorem that claims that, for a regular n-gon, all pairs of diagonals are

(1) either congruent

(2) or incommensurable if and only if 6 does not divide n; in this second case the diagonals d_1 and d_2 are commensurable if and only if $d_1 = 2d_2$, where d_1 is a diagonal of maximum length.

(For a proof see https://doi.org/10.1007/s00013-020-01477-w.) We are going to conclude that the only solution is $r_1 = r_2$ where both radii are the same (but arbitrary) integers. Little John's method produces an exact regular hexagon in such cases.

Let us assume that for a given pair of integer numbers r_1 and r_2, Little John's method produces a regular n-gon. We distinguish between two cases:

(a) n is even. In this case, r_1 is the half of the diameter which is a diagonal of the n-gon. Now, $2r_1$ and r_2 are the lengths of a pair of diagonals in the n-gon and they are commensurable. Here there are two cases, according to Vincenzi's theorem:

i. These diagonals are congruent. That is, $2r_1 = r_2$, but in this case Little John's method produces a "2-gon" which has no geometrical meaning.

ii. These diagonals are not congruent, n is a multiple of 6, and $2r_1 = 2r_2$. This implies $r_1 = r_2$ and Little John's method produces a regular hexagon.

(b) n is odd. In this case let us consider a regular $2n$-gon by extending the produced n-gon in such a way that they share the same circumcircle and every second vertex of the $2n$-gon is a vertex of the n-gon as well. Clearly, the radius of the circumcircle is $2r_1$. Therefore, in the $2n$-gon we found two commensurable diagonals of lengths $2r_1$ and r_2. According to Vincenzi's theorem, there are two cases:

i. these diagonals are congruent. That is, $2r_1 = r_2$, but in this case Little John's method produces a "2-gon" again, and this has no geometrical meaning.

ii. These diagonals are not congruent, $2n$ is a multiple of 6, and $2r_1 = 2r_2$. This implies $r_1 = r_2$ and Little John's method would produce a regular hexagon, but in this case n cannot be odd (so this case cannot occur).

At the end of the day, we can learn that only the case (a) ii. can occur. This confirms our statement: *Little John's method can construct only a regular hexagon in an accurate way, and this is the same as the well-known method that is widely used in schools as well.*