PROBLEM CORNER

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Problem 1 *We suppose that we have n white balls and n black balls which we are going to place in two urns A and B in any way we please, as long as at least one ball is placed into each urn. After this has been done, a second person walks into the room and selects one ball at random. Our problem is to maximize the probability that this person draws a white ball.*

Solution: We suppose that the distribution of the balls in the urns A and B is as described in the following table:

	A	B
Number of White Balls	x	$n-x$
Number of Black Balls	y	$n-y$

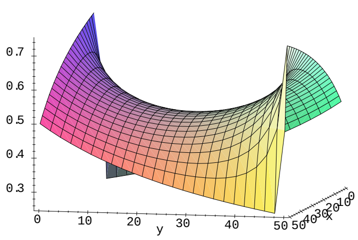
If $P(x, y, n)$ is the probability that a single ball drawn at random will be white then

$$P(x,y,n) = \frac{1}{2}\left(\frac{x}{x+y} + \frac{n-x}{2n-x-y}\right).$$

From now on we shall assume that $n = 50$. We begin our study of the function by looking at the following table which shows the values of $P(x, y, 50)$ at a few selected points (x, y).

$P(0,1,50) = .25253$	$P(1,0,50) = .74747$
$P(1,1,50) = .5$	$P(2,1,50) = .58076$
$P(1,2,50) = .41924$	$P(25,25,50) = .5$
$P(50,1,50) = .4902$	$P(1,50,50) = .5098$
$P(50,0,50) = .4902$	$P(50,49,50) = .25253$
$P(49,50,50) = .74747$	$P(49,49,50) = .5$

To solve the problem we need to find the maximum value of the expression $P(x, y, 50)$ as the point (x, y) varies through the rectangle $[0, 50] \times [0, 50]$ from which the points $(0, 0)$ and $(50, 50)$ have been removed. If we sketch the graph $z = P(x, y, 50)$ then we obtain the following surface



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From the looks of this surface it seems unlikely that the maximum value of z will be achieved at a critical point. The maximum appears to be at the left or right extremities of the figure. As a matter of fact, if we point at the equations

$$\begin{aligned} \frac{\partial}{\partial x}P(x,y,50) &= 0 \\ \frac{\partial}{\partial y}P(x,y,50) &= 0 \end{aligned}$$

and after solvving these equations, we obtain

$$\{y = 25, x = 25\}.$$

As we have already seen, the maximum value of z does not occur at the point $(25, 25)$.

We now examine the boundary behavior of the function. There are four cases to consider

The Case $x = 0$ and $1 \le y \le 50$ We define $g(y) = P(0, y, 50)$ for $1 \le y \le 50$.Since

$$g(y) = \frac{25}{100 - y}$$

for each y we see that the maximum value of $g(y)$ is $g(50) = \frac{1}{2}$.

The Case $y = 0$ and $1 \le x \le 50$ We define $g(x) = P(x, 0, 50)$ for $1 \le x \le 50$. Since

$$g(x) = \frac{1}{2} + \frac{1}{2}\frac{50 - x}{100 - x} = 1 - \frac{25}{100 - x}$$

for each x, we see that the maximum value of this function is $g(1) = .74747$.

The Case $x = 50$ and $0 \le y \le 49$ We define $g(y) = P(50, y, 50)$ for $0 \le y \le 49$. Since

$$g(y) = \frac{25}{50 + y}$$

for each y, we see that the maximum value of this function is $g(0) = .5$.

The Case $y = 50$ and $0 \le x \le 49$ We define $g(x) = P(x, 50, 50)$ for $0 \le x \le 49$. Since

$$g(x) = 1 - \frac{50}{x + 50}$$

for each x, we see that the maximum value of this function is $g(49) = .74747$.

Some of the variations suggested here may be suitable for presentation in the classroom. Others may be suitable for student projects [1].

1. Repeat the preceding probability problem assuming that the selection of the ball will be made in such a way that the probability that the selection will be made from urn A is $\frac{1}{3}$ and the probability that the selection will be made from urn B is $\frac{2}{3}$.

2. Extend the preceding variation to the general case in which the probability of selecting a ball from urn A is some number γ satisfying $0 < \gamma < 1$ and the probability of selecting the ball from urn B is $1 - \gamma$.

3. Investigate the problem of determining how the balls should be placed in order to minimize the probability that the selected ball be white. Of course, this is simply the problem of maximizing the probability that a black ball be selected.

4. Suppose that the selection of the ball results in payoffs as described in the following table

	From urn A	From urn B
white ball	α_A	α_B
black ball	β_A	β_B

Study the payoffs that result from different placement of the balls.

5. Determine the maximum value of the expression $P(x, k, n)$ where k is a given integer satisfying $1 \le k \le n-1$. Find the value x_k of x at which this maximum occurs. You will find that

$$x_k = \frac{\sqrt{k}(2n-k) - k\sqrt{n-k}}{\sqrt{n-k} + \sqrt{k}}$$

and that the maximum value of $P(x, k, n)$ is

$$P(x_k, k, n) = \frac{3n - 2\sqrt{(k\,(n-k))}}{4n}$$

The next Problem can be done by using the Green's Theorem when one learns the multivariable calculus. Try to think how you may solve the problem without using the Green's Theorem.

Problem 2 *(See [2]) Suppose we are given two curves, one is cardioid of* $C_1 = [2\cos(t) - \cos(2t), 2\sin(t) - \sin(2t)]$ *(shown in green in Figure below), where* $t \in [0, 2\pi]$*, and the circle* $C_2 = [0.5 + 3\cos(t), 0.5 + 3\sin(t)]$*, (shwon in red in Figure below), where* $t \in [0, 2\pi]$*. We shall find the area bounded by the following area: Start at the point* B *on the green curve* C_1 *until the point* A *and then follow the red curve* C_2 *back to the point* B*.*

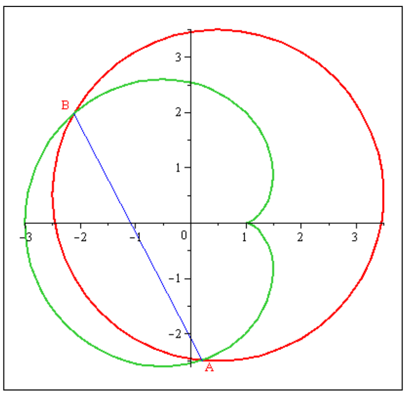

Figure. Intersection between C_1 *and* C_2*.*

Solution:

We need some background information that can be find the area bounded by a curve and a slanted line, see [2]. We state the theorems as follows:

Theorem 3 *Let C be the smooth curve, $\mathbf{w}(t) = [x(t), y(t)]$, where $t_1 \le t \le t_2$. Let R be the region bounded by C, by the line $y = mx + b$ and by the perpendiculars to the line from $(x(t_1), y(t_1))$ and $(x(t_2), y(t_2))$. Then the area of R is given by*

$$\frac{1}{1+m^2}\int_{t_1}^{t_2} (-x(t)m + y(t) - b)\,(x'(t) + y'(t)m)\,dt. \tag{1}$$

The Green's Theorem in our discussion can be summarized as follows:

Theorem 4 *Let C be the smooth curve, $\mathbf{w}(t) = [x(t), y(t)]$, where $t_1 \le t \le t_2$. Let R be the region bounded by C, by the line $y = mx + b$ and by the perpendiculars to the line from $(x(t_1), y(t_1))$ and $(x(t_2), y(t_2))$. If $P(x,y) = \frac{y}{2}$ and $Q(x,y) = \frac{x}{2}$, then*

$$\begin{aligned}\int_C P\,dx + Q\,dy &= \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\ &= \frac{1}{1+m^2}\int_{t_1}^{t_2} (-x(t)m + y(t) - b)\,(x'(t) + y'(t)m)\,dt.\end{aligned} \tag{2}$$

1. Step 1. First, we find the intersections for these two curves at

   $$A = (.2046062939, -2.485421672) \text{ and } B = (-2.108615230, 1.981595956)$$

   respectively.

2. Step 2. We calculate the area bounded by C_1 and line segment AB.If we travel along C_1 from B to A, the angle is from $t = 4.402664587$ to $t = 2.574088482$.We apply the area formula to compute the area bounded C_1 and the line equation AB :

   $$y = -1.931x - 2.09,$$

   and obtain

   $$\begin{aligned}&\frac{1}{1+m^2}\int_{t_1}^{t_2} |(-x(t)m + y(t) - b)\,(x'(t) + y'(t)m)|\,dt \\ =\ &\frac{1}{1+(-1.931)^2}\int_{4.402664587}^{2.574088482} \begin{matrix}((\cos 2t - 2\cos t)(-1.931) + 2\sin t - \sin 2t + 2.09)\cdot \\ (-2\sin t + 2\sin 2t + (2\cos t - 2\cos 2t)(-1.931))\end{matrix}\,dt \\ =\ &7.538433161.\end{aligned} \tag{3}$$

3. Step 3. We next calculate the area bounded by C_2 and the line segment AB. If we travel along C_2 from A to B,the angle is from $t = 2.625063229$ to $t = 4.613764607$.We apply the area formula to compute the area bounded C_1 and the line equation AB :

   $$y = -1.931x - 2.09,$$

and obtain

$$\begin{aligned}
& \frac{1}{1+m^2}\int_{t_1}^{t_2}\left|(-x(t)m+y(t)-b)\left(x'(t)+y'(t)m\right)\right|dt \qquad (4)\\
= & \frac{1}{1+(-1.931)^2}\int_{2.625063229}^{4.613764607} \begin{array}{c}((-0.5-3\cos t)(-1.931)+0.5+3\sin t+2.09)\cdot \\ (-3\sin t+3\cos t)(-1.931))\end{array} dt \\
= & -4.836961037.
\end{aligned}$$

Therefore, the total net area bounded by C_1 and C_2 is 2.701472124.

Remark 5 *A quick check using Green's Theorem on the curve* $C = C_1^* \cup C_2^*$,*where* C_1^* *is when we travel along* C_1 *from* B *to* A *and* C_2^* *is when we travel along* C_2 *from* A *to* B*, we get the answer of* 2.701472124.

$$\begin{aligned}
& \frac{1}{2}\int_{C_1^*}\left(x(t)y'(t)-y(t)x'(t)\right)dt+\frac{1}{2}\int_{C_2^*}\left(x(t)y'(t)-y(t)x'(t)\right)dt \\
= & -9.955568535+7.254096410=-2.701472125.
\end{aligned}$$

We note that the Green' Theorem produces a negative area if the curve travels in clock-wise direction.

References

[1] Yang, W.-C., Lin, C.-C. & Thompson, S. "*New Approaches to a Classical Problem*", International Journal of Mathematical Education in Science and Technology, 1998, Vol. 29, No. 4, pp613-617.

[2] Yang, W.-C. & Lo, M.-L. "*Finding signed Areas and Volumes inspired by Technology*", Electronic Journal of Mathematics and Technology (eJMT), ISSN 1933-2823, Issue 2, Vol. 2, June 2008.