PROBLEM CORNER

Problem 1.

Let ABC be a triangle. For any point P, call Oa, Ob, Oc the circumcenters of triangles PBC, PCA, PAB, respectively (see Figure 1). Find the locus of P such that triangles $OaObOc$ and ABC have the same area.

(proposed by Francisco Javier GARCÍA CAPITÁN, garciacapitan@gmail.es, I.E.S. Álvarez Cubero, Priego de Córdoba, SPAIN)

Solution (by Francisco Javier GARCÍA CAPITÁN)

We use *Mathematica*[1] and *Baricentricas*[2] package with the following commands:

```
ptP = {x, y, z};
{ptOa, ptOb, ptOc} = Map[Circuncentro,
{{ptP, ptB, ptC},
{ptP, ptC, ptA},
{ptP, ptA, ptB}}];
locus = Numerator[Factor[AreaTriangulo[
{ptOa, ptOb, ptOc}] + 1]]
```

In this way we get the quartic

$$\sum_{\text{cyclic}} \left(2\left(S^2 + S_A^2\right) + a^4 yz\right) yz = 0,$$

whose isogonal transform has equation

$$\begin{aligned} &4S^2(a^2yz + b^2zx + c^2xy) + a^2b^2c^2(x+y+z)^2 = 0 \\ \Leftrightarrow\ &a^2yz + b^2zx + c^2xy + R^2(x+y+z)^2 = 0. \end{aligned}$$

This is the circle with center O and squared radius $2R^2$, that is, the image of the circumcircle by a homothety with center O and ratio $\sqrt{2}$.

[1] https://www.wolfram.com/mathematica/

[2] See https://garciacapitan.epizy.com/baricentricas/?i=1 for information and links to download this *Mathematica*-based package developed by García Capitán.

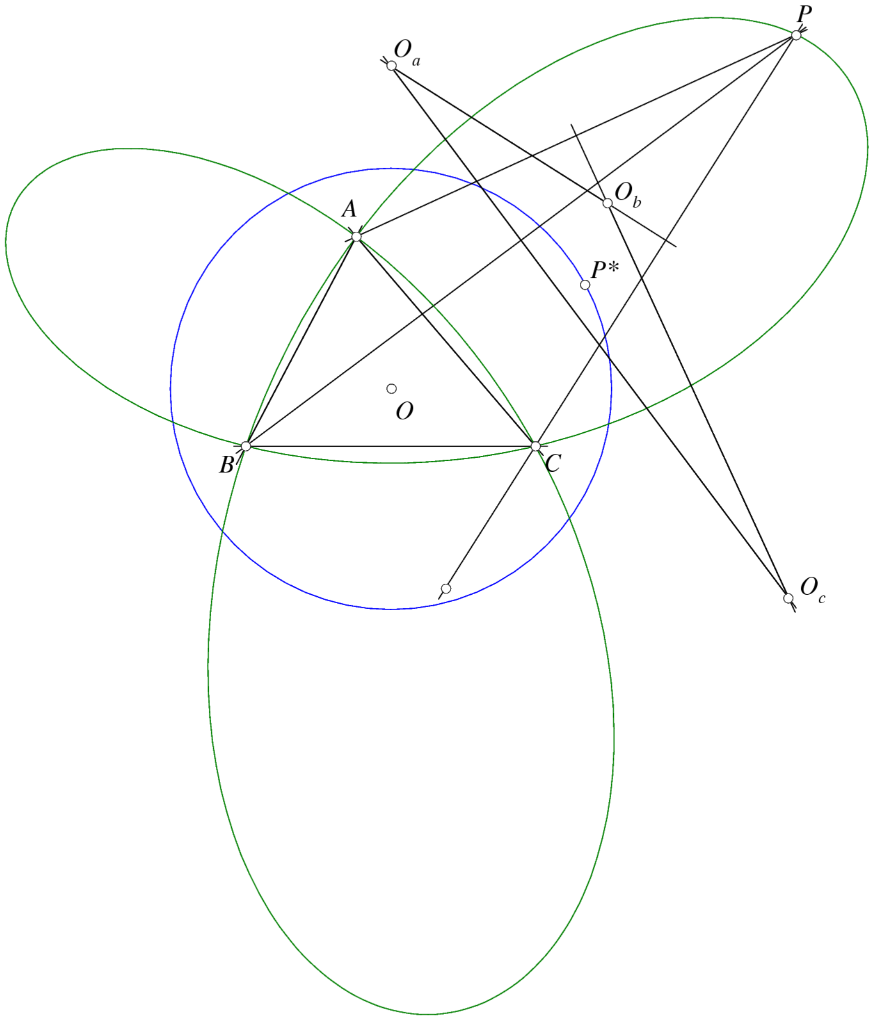


Figure 1

Remark. Observe that we have looked for $(O_aO_bO_c) = -(ABC)$. If we had tried $(O_aO_bO_c) = (ABC)$ we had got that the isogonal transform of our locus is the circle with center O and radius 0, then the only point satisfying the property is the circumcenter.

Problem 2.

Let ABC be an equilateral triangle, and let Ω be its inscribed circle of radius r. Let P be an arbitrary point of this circle. Let us construct the equilateral triangles (vertices clockwise) PAD, PBE, PCF, and APM, BPN, CPO. Prove that the triangles DEF and MNO are similar to ABC. Prove that the circles circumscribed to DEF and MNO are tangent to Ω, have their centers in Ω and their radii are $2r$.

(proposed by Ricardo BARROSO CAMPOS, rbarrosoenator@gmail.com, Sevilla, SPAIN)

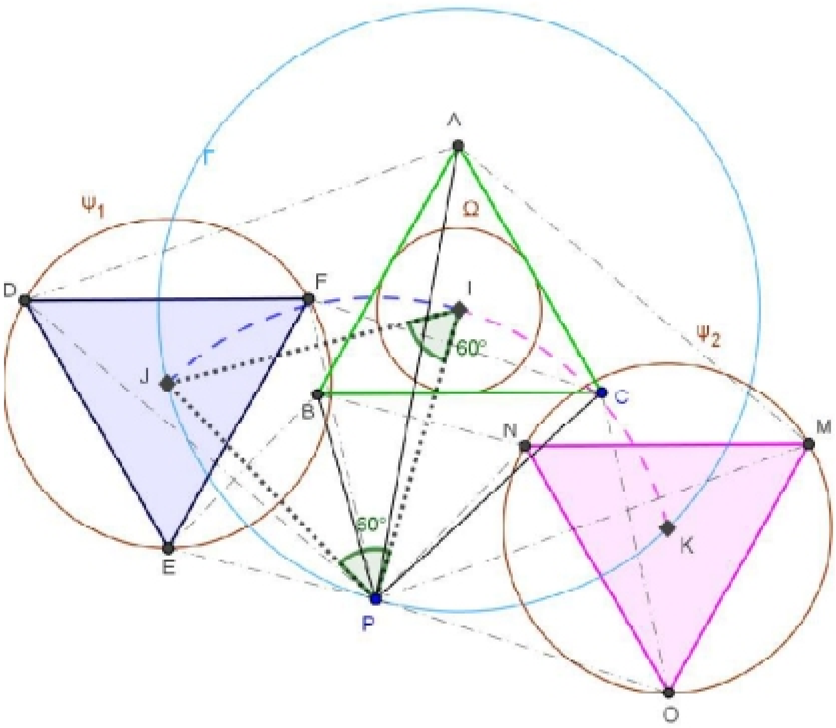
The above problem was initially proposed in the *Laboratorio Virtual de Triángulos Cabri*, an electronic magazine managed by Ricardo Barroso along the years 2000-2023, that every two weeks presented contributions from different authors concerning new elementary geometry problems, as well as solutions emphasizing the role of dynamic geometry software such as Cabri[3]. See https://trianguloscabri.github.io/ for a short description of this *Laboratorio*... including links that contains all the different issues of the *Laboratorio*...

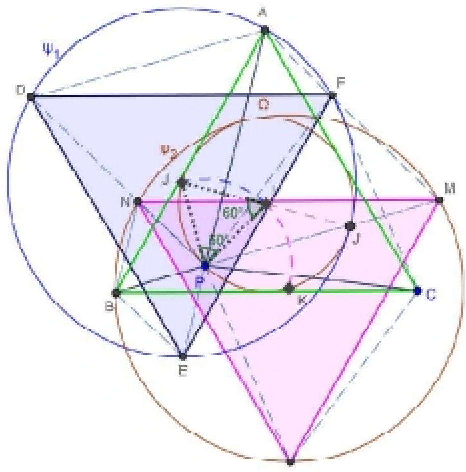
Thus, Problem 2 was proposed in the *Laboratorio* by the own editor of the journal, Ricardo Barroso, back in 2022, as *Problem 1056*, and received several solutions (three in Spanish and one in French). To find them, look for the last problem in https://trianguloscabri.github.io/curso20212022/index.htm that includes links to the solutions. Since the solutions appear as pdf files, we do not dare to include them here, but we strongly suggest the readers to look there for the different and beautiful solutions. Yet, we have received from prof. Barroso, the Word version of one of these solutions, that we have translated and is included next.

Solution (by SATURNINO CAMPO-RUIZ, Mathematics Teacher, retired, from Salamanca, Spain)

Although the problem seems much more complex, we are only making a couple of rotations (of amplitudes ±60°) of the equilateral triangle ABC, with P as the center of the rotation. For all these reasons, the triangles DEF and MNO with centers J and K respectively, are equilateral, with the same side as ABC and parallel sides. The center of each of them is the center I of ABC transformed by the rotation; therefore, the circumscribed circles of the triangles ABC, EF and MNO have the same radius, which is twice the radius of the inscribed circle.

[3] https://cabri.com





Furthermore, triangles PIJ, PIK are equilateral since $PI = PJ = PK$ and $\measuredangle JPI = 60° = \measuredangle IPK$. Therefore J and K and P itself are located on the circle Γ with center I and radius IP. Let ψ_1 and ψ_2 be the circles circumscribed to DEF and MNO. If P is on Ω (inscribed circle), the centers J of ψ_1 and ψ_2 are also on Ω (coincides with Γ). The symmetric J' of J with respect to I is on Ω and JJ' is the radius of ψ_1. Therefore these circles are tangent at that point. And likewise for ψ_2 and Ω.

Alternative Solution (by TOMAS RECIO, trecio@nebrija.es, Universidad Nebrija, Madrid, Spain)

Let us just remark that, using the Automated Reasoning Tools (ART) of GeoGebra Discovery https://github.com/kovzol/geogebra-discovery?tab=readme-ov-file , see also Kovács et al. 2020, 2022) we can check immediately (and with mathematical rigor, although without offering a human-readable proof) that triangles *DEF* and *MNO* are similar to *ABC*. See Figures 2, 3 and 4, declaring, by answering to the *Relation(,)* command, involving in each case two segments, that the corresponding sides of the triangles are equal. Moreover, the output of Figure 5 shows, using the Discovery( ) command, that outputs diverse results involving a selected point, that both triangles are similar to *ABC*.

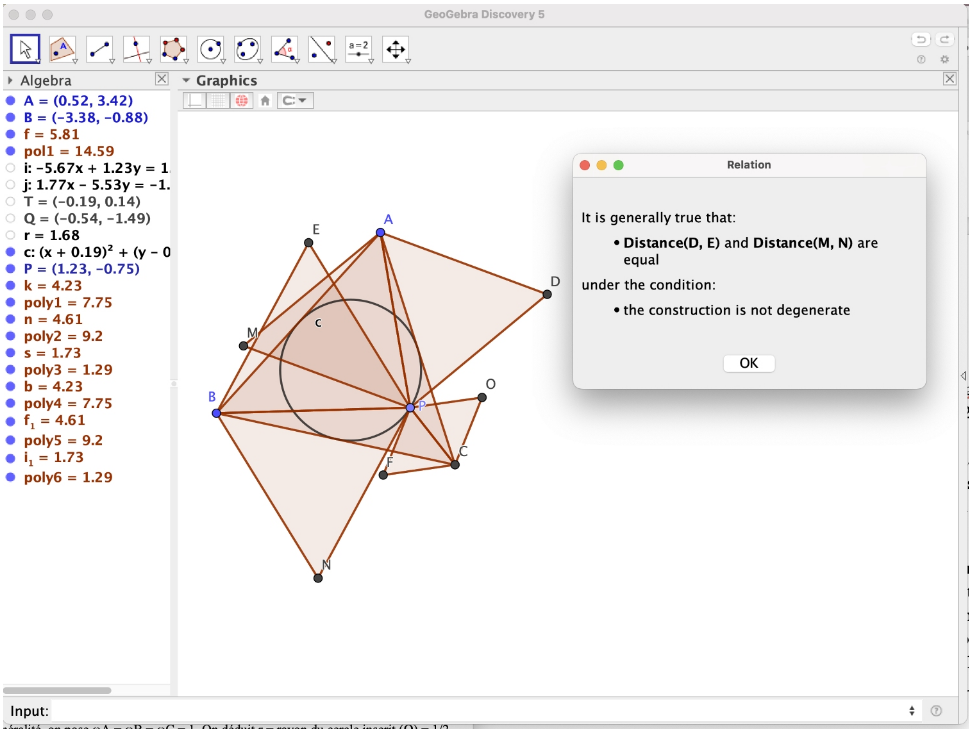


Figure 2: GeoGebra Discovery Relation(DE, MN) command answers that DE=MN

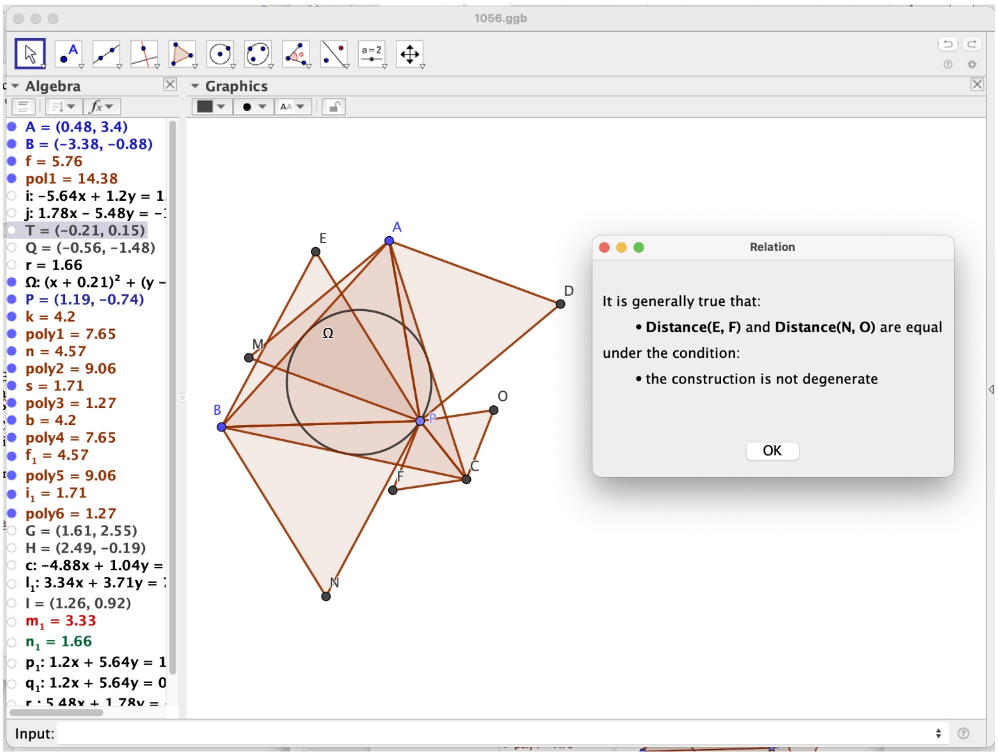


Figure 3: GeoGebra Discovery states that EF=NO

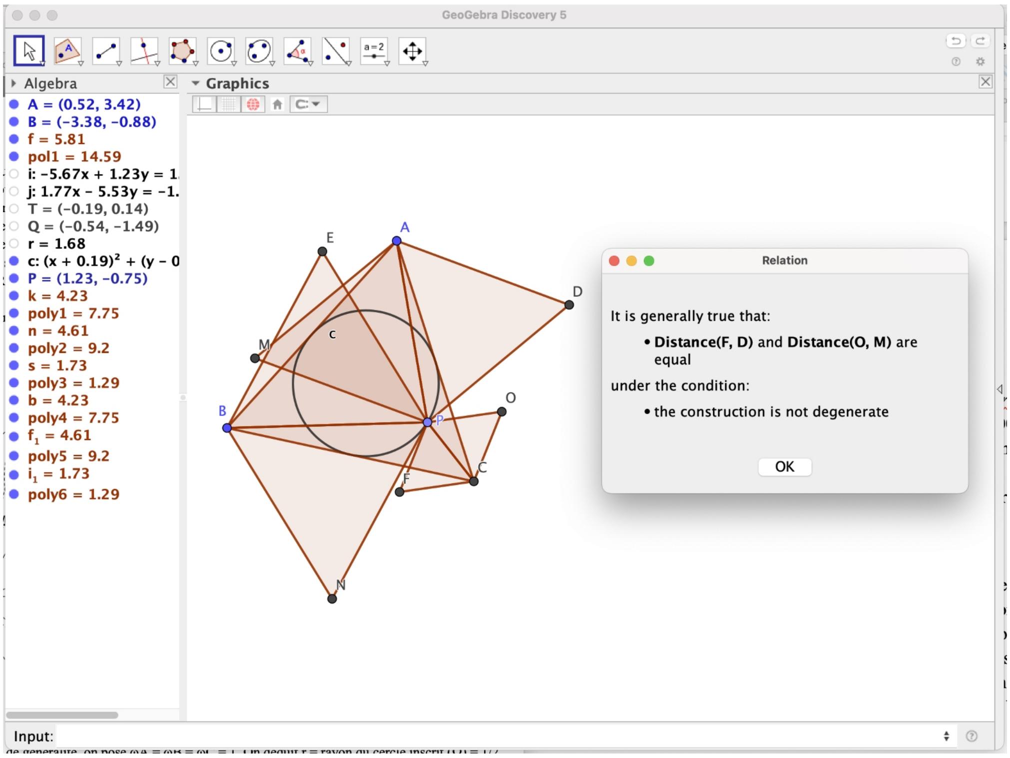


Figure 4: GeoGebra Discovery states that FD=OM

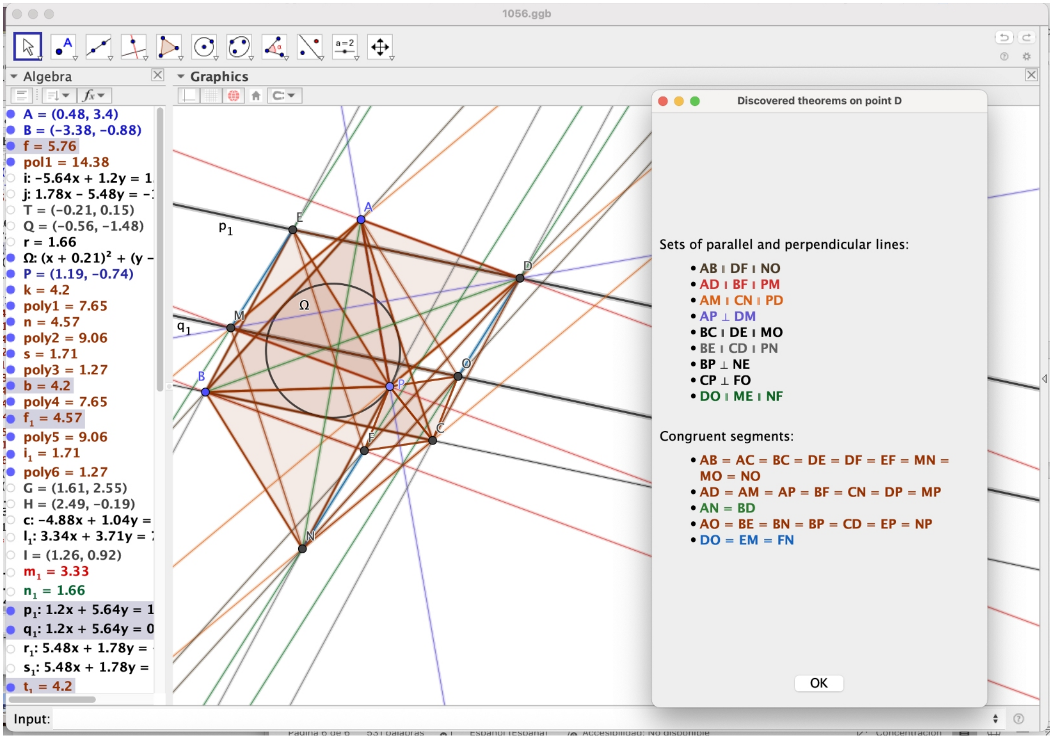


Figure 5: Using GeoGebra Discovery Discover(D) *command, one obtains several results involving point D, in particular, the equality of lengths of the sides of DEF and MNO with AB*

As a consequence, since for equilateral triangles it holds that the incircle radius is half

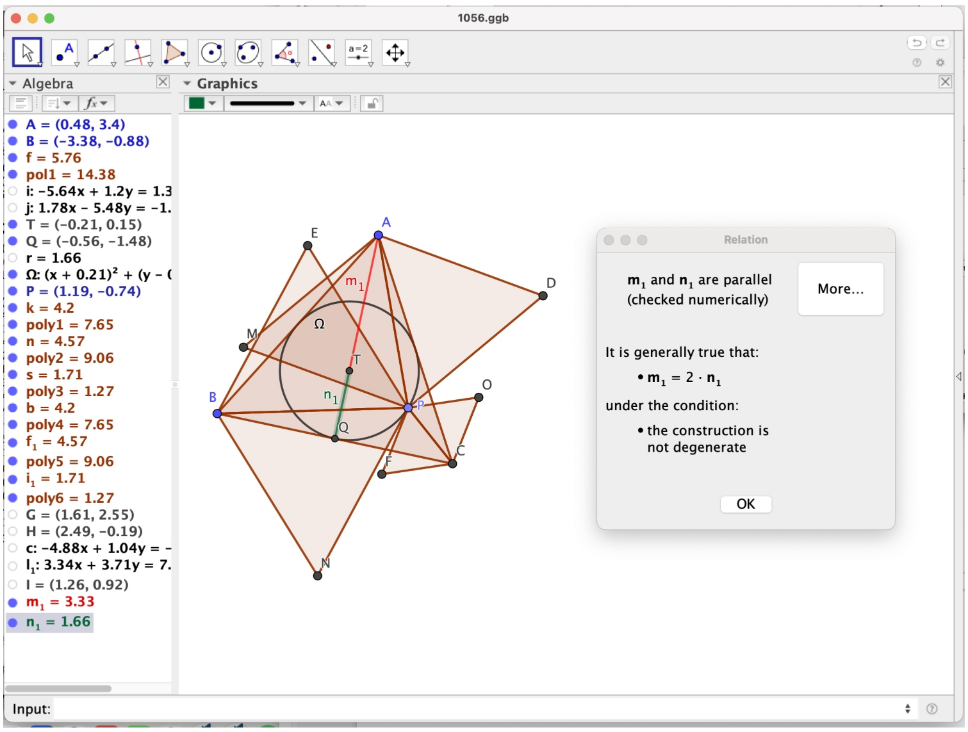


Figure 6: The radius n_1 of the incircle is half of the radius m_1 of the circumcircle

the circumcircle radius, we conclude that the circles circumscribed to *DEF* and *MNO* have radii *2r*, where *r* is the radius of the incircle of *ABC*. See Figure 6, describing the relation between the radius n_1, as a segment from the incenter *T* of triangle *ABC* (that coincides with the baricenter, for equilateral triangles) to the feet *Q* of the perpendicular from *T* to side *BC*, and the radius m_1, from *T* to vertex *A*.

Finally, we have to confess that, currently, the tangency of the circumcircles of *DEF* and *MNO* to the incircle of *ABC*, and the fact that their centers lie in such incircle, is a statement too involved for being directly handled by GeoGebra Discovery. But GeoGebra Discovery can help substantially: one can check –by dealing, through the Relation (,) command, with the lengths of the sides of triangle *AMP*, and the same for *BPN, CPO*, etc-- that triangle *MNO* is the rotation of *ABC* around *P* by 60º (counterclockwise). See Figure 7.

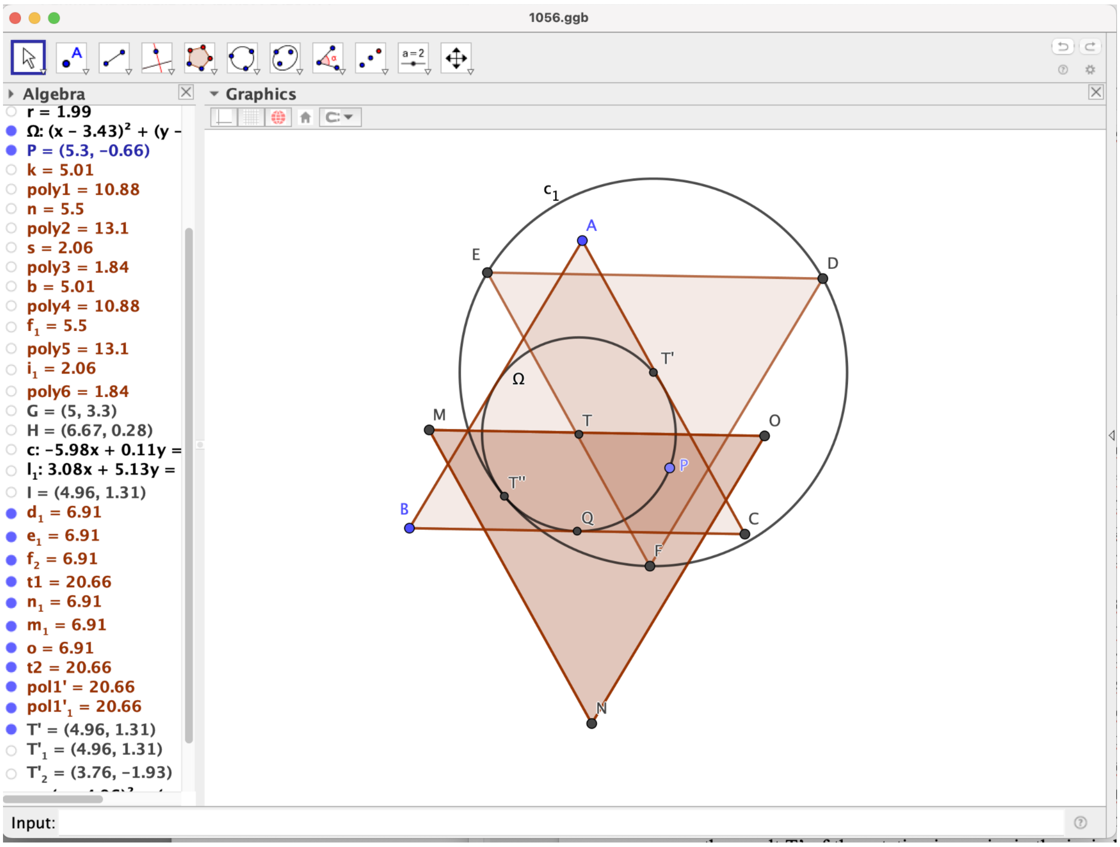


Figure 7: Tangency of the incircle and circumcircle

Likewise, for triangle *DEF*, by 60º (clockwise). Thus, their centers will be also the result of rotating the incenter of *T* around *P* by the corresponding degrees. But such rotation, yielding, for example, point *T'*, with two equal sides *TP=T'P*, and angle 60º in between, keeps lenghts *TP=TT'*, so the result *T'* of the rotation is, again, in the incircle. Moreover, bearing in mind this lengh and the perpendicularity of *AT* to *BC*, that yields the perpendicularity of *DT'* to *EF*, it is easy to show that *T'* will be the center of the circumcircle of *DEF*. Finally, notice that the symmetrical *T''* of *T'* with respect to *T* will lie also in the incircle and the distance $T'T''$ will be $2r=T'T+TT''$. So T'' will lie in the circumcircle, and will be the common tangency point, as the incircle will be always inside the circumcircle when the triangle is rotated.

References

Kovács Z.; Recio T. (2020). GeoGebra reasoning tools for humans and for automatons. *Electronic Proceedings of the 25th Asian Technology Conference in Mathematics*, December 14-16, 2020. ISSN 1940-4204 http://atcm.mathandtech.org/EP2020/invited/21786.pdf

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