

Problem Corner

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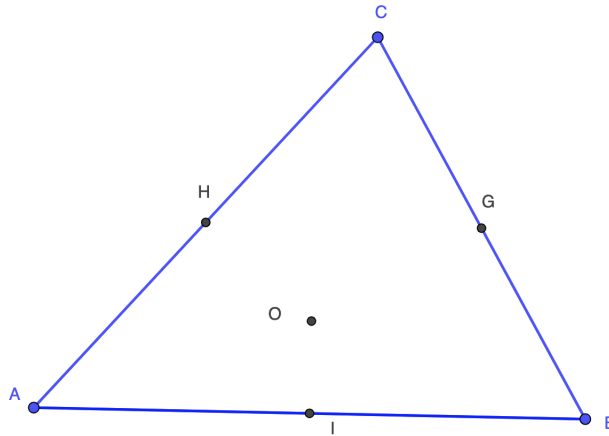
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Problem 1

Let $[ABC]$ be any triangle and let D , E , and F be the reflections of the circumcentre in relation to the three sides. Show that triangles $[ABC]$ and $[DEF]$ are congruent¹.

Solution

Let O be the point of intersection of the perpendicular bisectors, and let G , H and I be the points where the perpendicular bisectors intersect the sides $[BC]$, $[AC]$ and $[AB]$, respectively.



The triangle $[AIH]$ is similar to $[ABC]$ by the SAS (side-angle-side) criterion, since they share the angle at A and

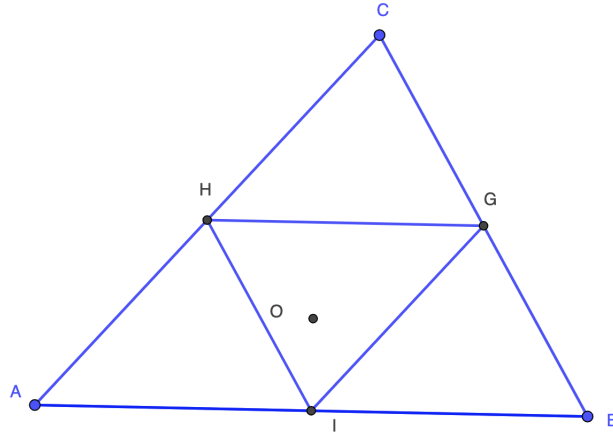
$$\frac{AI}{AB} = \frac{AH}{AC} = \frac{1}{2},$$

because I and H are midpoints of the sides. Similarly, one may justify the similarity of triangles $[HGC]$ and $[IBG]$ with triangle $[ABC]$.

From these three similarities it follows that

$$\frac{HI}{CB} = \frac{IG}{AC} = \frac{HG}{AB} = \frac{1}{2},$$

¹Portuguese Olympiad Math problem (2018)



therefore, by the SSS (side-side-side) criterion, triangle $[GHI]$ is similar to triangle $[ABC]$ with similarity ratio $\frac{1}{2}$.

Hence, the triangles $[AIH]$, $[IBG]$, $[HGC]$ and $[GHI]$ are congruent, since they are all similar to triangle $[ABC]$ with similarity ratio $\frac{1}{2}$.

Now considering triangles $[HIG]$ and $[EDF]$, since

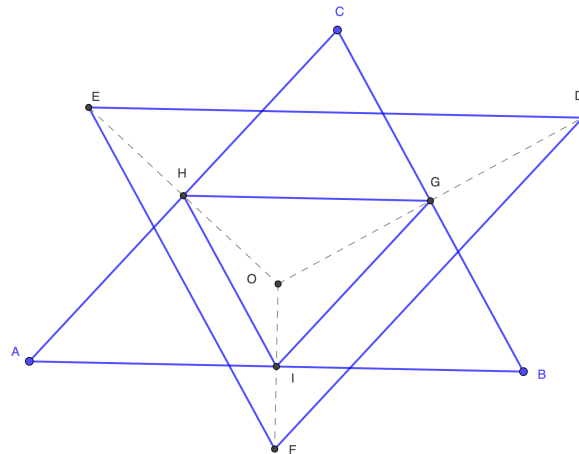
$$OI = IF, \quad GD = OG \quad \text{and} \quad OH = HE,$$

there are three pairs of similar triangles by the SAS criterion: $[OHI]$ and $[OEF]$, $[OIG]$ and $[OFD]$, $[OHG]$ and $[OED]$, all sharing the corresponding angle at O , and with similarity ratio equal to $\frac{1}{2}$.

Finally, we obtain

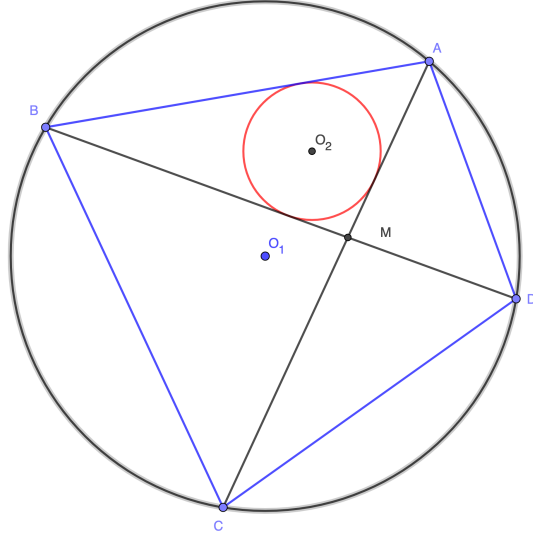
$$ED = 2HG = AB, \quad EF = 2HI = CB \quad \text{and} \quad FD = 2IG = AC,$$

therefore triangles $[ABC]$ and $[DEF]$ are congruent by the SSS criterion.



Problem 2

Let the quadrilateral $[ABCD]$ be inscribed in a circle with centre R , and let a circle with centre r be inscribed in triangle $[ABM]$, where M is a point on segment BD . Suppose that the product $Rr = A$, and that the side lengths are $AD = a$, $BC = b$, and $AB = c$. Determine the length of segment $CD = x$ in terms of A , a , b , and c^2 .



Solution

Let $\angle BCD = \alpha$, then, $\angle BAD = 180^\circ - \alpha$.

In $\triangle BCD$ and $\triangle BAD$,

$$BD^2 = x^2 + b^2 - 2bx \cos \alpha \quad (1)$$

$$\begin{aligned} BD^2 &= a^2 + c^2 - 2ac \cos(180^\circ - \alpha) \\ &= a^2 + c^2 + 2ac \cos \alpha \end{aligned}$$

$$\therefore x^2 + b^2 - 2bx \cos \alpha = a^2 + c^2 + 2ac \cos \alpha$$

$$\therefore \cos \alpha = \frac{x^2 + b^2 - a^2 - c^2}{2(bx + ac)}$$

Substituting this into (1), and simplifying:

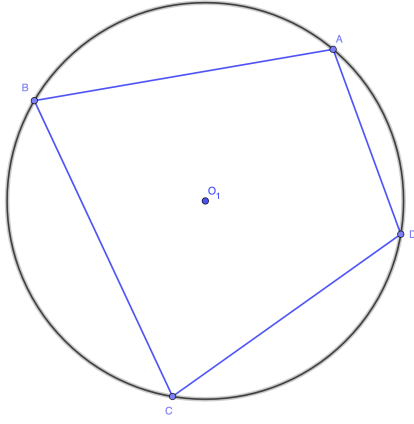
$$BD^2 = \frac{(ax + bc)(cx + ab)}{bx + ac} \quad (2)$$

From $\triangle DMC \sim \triangle AMB$,

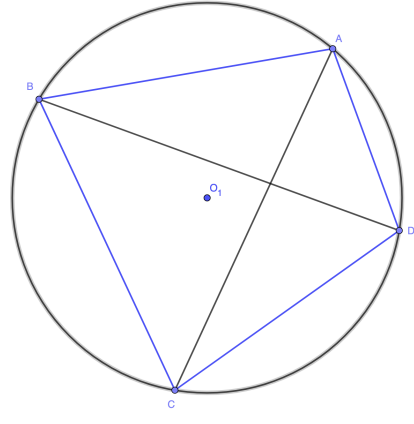
$$\frac{x}{DM} = \frac{c}{AM}$$

$$\therefore DM = \frac{x}{c} AM \quad (3)$$

²Japanese temple mathematical problems in Nagano Pref. Japan, 2003



Step 1



Step 2

Similarly,

$$BM = \frac{b}{a}AM \quad (4)$$

From (3), (4),

$$BD = DM + BM = \left(\frac{x}{c} + \frac{b}{a} \right) AM$$

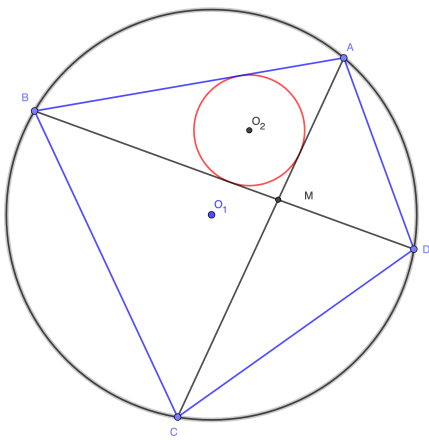
$$\therefore AM = \frac{ac}{ax + bc} BD$$

From (2),

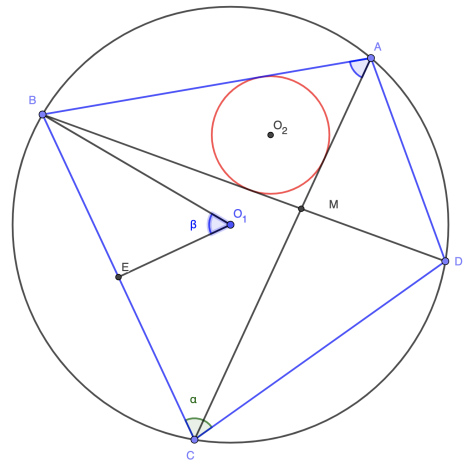
$$AM = \frac{ac\sqrt{cx + ab}}{\sqrt{ax + bc}\sqrt{bx + ac}}$$

Similarly,

$$BM = \frac{bc\sqrt{cx + ab}}{\sqrt{ax + bc}\sqrt{bx + ac}}$$



Step 3



Step 4

Let $\angle BO_1E = \beta$, then, $\angle BAC = \beta$, and $\sin \beta = \frac{b/2}{R} = \frac{b}{2R}$

About the area of $\triangle ABM$,

$$\frac{1}{2}AM \cdot AB \sin \beta = \frac{1}{2}r(AM + BM + AB)$$

$$\therefore 2AM \cdot c \cdot \frac{b}{2R} = 2r(AM + BM + c)$$

Therefore,

$$4rR = \frac{2bc \cdot AM}{AM + BM + c}$$

$$A = \frac{abc}{2 \left(a + b + \sqrt{\frac{(ax + bc)(bx + ac)}{cx + ab}} \right)}$$

$$\sqrt{\frac{(ax + bc)(bx + ac)}{cx + ab}} = \frac{abc}{2A} - (a + b)$$

Square both sides, and put x in order:

$$x^2 - 2c \left(1 - \frac{c}{2A} \left(a + b - \frac{abc}{4A} \right) \right) x + c^2 - \left(\frac{abc}{2A} - (a + b) \right)^2 = 0$$

$$\therefore x = c \left(1 - \frac{c}{2A} \left(a + b - \frac{abc}{4A} \right) \right) + \sqrt{c^2 \left[1 - \frac{c}{2A} \left(a + b - \frac{abc}{4A} \right) \right]^2 - c^2 + \left(a + b - \frac{abc}{2A} \right)^2}$$

(*) In the same way, $AC^2 = \frac{(bx + ac)(cx + ab)}{ax + bc}$. Therefore, the Ptolemy theorem is obtained: $AC \cdot BD = cx + ab$