Solutions to Problems from October 2011

Question 1 (2007, Jiangsu, China) Let $f(x) = \log_2[2x^2 + (m+3)x + 2m]$. It is known that $f(x) \in (-\infty, \infty)$. Find the range of the parameter m. Extend your results to solving similar problems.

Solution. In Figure 1 below, the dashed curve shows that there is no point of intersection of the x-axis and the curve $g(x) = 2x^2 + (m + 3)x + 2m$ when $m = 1.5$. So for $m = 1.5$, let t_0 be the point such that $g(t_0)$ is the absolute minimum which is positive. Then the range of the function $f(x)$ is $[f(t_0), \infty)$, which is a subset of ℝ but not equal to ℝ.

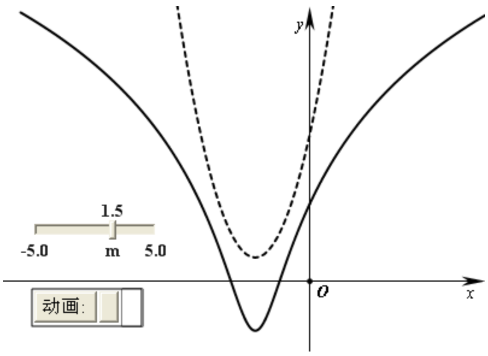

Figure 1

In Figure 2, the dashed curve shows that there is only one intersection point of the x-axis and the curve $g(x) = 2x^2 + (m+3)x + 2m$ when $m = 1.0$. In this case, the function $g(x)$ has the minimum value 0. So the values of $g(x)$ need to be positive in order to make the function $f(x)$ well defined. Therefore the range of the function $f(x)$ is ℝ when $m = 1.0$.

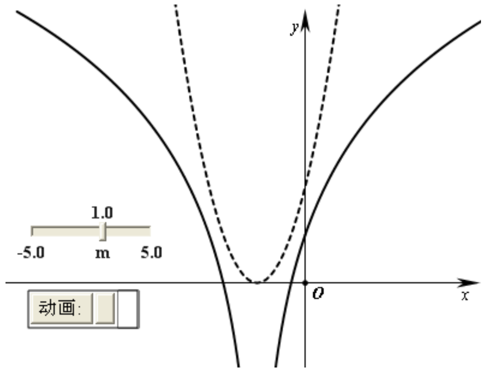

Figure 2

In Figure 3, the dashed curve shows that there is only one intersection point of the x-axis and the curve $g(x) = 2x^2 + (m+3)x + 2m$ when $m = -0.7$. So when $m = -0.7$, the absolute minimum of $g(x)$ is a negative number. As before, the values of $g(x)$ need to be positive to make the function $f(x)$ well defined. Therefore the range of the function $f(x)$ is ℝ when $m = -0.7$.

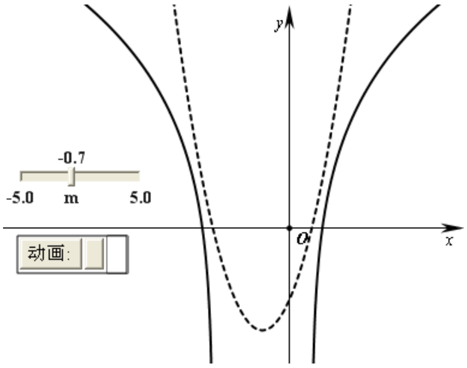

Figure 3

From the discussion above， we can see that that the range of the function $f(x)$ is $\mathbb{R}$ only when there is/are intersection point(s) of the x-axis and the curve $g(x) = 2x^2 + (m+3)x + 2m$ or, in other words, when the discriminant $\Delta = (m+3)^2 - 4 \times 2 \times 2m \geq 0$, or in other words when $m \leq 1$ or $m \geq 9$, the range of the function $f(x)$ is $\mathbb{R}$.

Question 2 (2009, Beijing, China) Let the point P be on the line $l: y = x - 1$, and let A and B be the two points of intersection of the parabola $y = x^2$ and a straight line passing through P. We call P an $\mathscr{A}$-point if $|PA| = |AB|$. Which one of the following statements is correct?

A. All of the points on the line $l: y = x - 1$ are $\mathscr{A}$–points.
B. Only finitely many points on line $l: y = x - 1$ can be called $\mathscr{A}$–points.
C. Not all of the points on line $l: y = x - 1$ are $\mathscr{A}$–points.
D. Infinitely many points, but not all of the points, on line $l: y = x - 1$ can be called $\mathscr{A}$–points.

Solution. For any point P on the line $y = x - 1$, we randomly draw a line passing through P. If the line intersects the parabola $y = x^2$, we define the intersection point closer to P as point A, as shown in Figure 4 below. Define B to be the point on line PA with $|PA| = |AB|$.

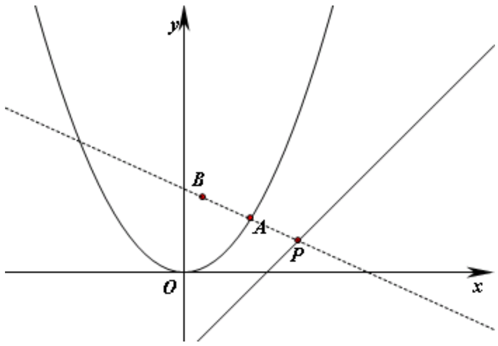

Figure 4

For any point P on the straight line $y = x - 1$, as shown in Figure 4, there must exist a point A as defined above such that point B is "above" the parabola, or B is in the region $y > x^2$ as shown in Figure 4.

On the other hand, there must also exist another point A with the corresponding point B "under" the parabola, or in other words located in the region $y < x^2$ as shown in Figure 5.

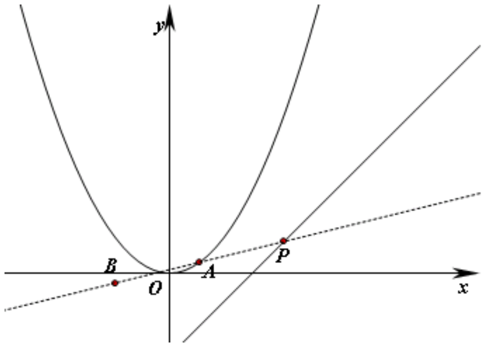

Figure 5

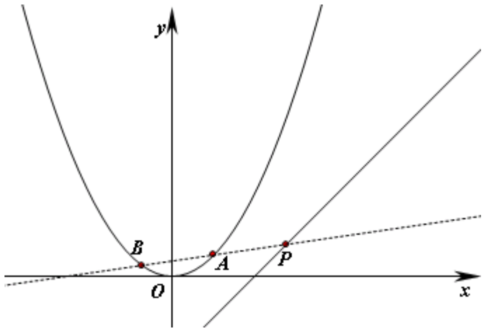

Figure 6

Since the parabola is a continuous curve, for any point P on the straight line $y = \text{x} - 1$, there must exist a point A on the parabola $y = x^2$ which makes the point B settle down exactly on the parabola as shown in Figure 6.

Therefore, the correct answer is "A".

Question 3 (2006, Hubei China) Consider the following four statements about the equation $(x^2 - 1)^2 - |x^2 - 1| + k = 0$ with variable x and constant k.

(i) There is a real number k that makes the equation have two different real roots.
(ii) There is a real number k that makes the equation have four different real roots.
(iii) There is a real number k that makes the equation have five different real roots.
(iv) There is a real number k that makes the equation have eight different real roots.

How many of these statements are false?

A. 0
B. 1
C. 2
D. 3

Solution. Let $y = |x^2 - 1|$. Then y is the root of the equation $x^2 - x + k = 0$. First of all, we discuss how many roots the equation $y = |x^2 - 1|$ has as y varies.

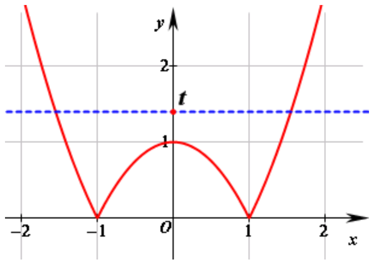

Figure 7

Figure 7 shows two curves: the solid one represents the function $y = |x^2 - 1|$ and the dotted line is the graph of function $y = t$. So the number of the intersection point(s) of the two curves illustrates the number of the roots of the equation $y = t$. So we have the following cases.

Case 1 When $t > 1$, the equation $y = t$ has two different real roots as in Figure 7.

Case 2 When $t = 1$, the equation $y = t$ has three different real roots as in Figure 8.

Case 3 When $0 < t < 1$, the equation $y = t$ has four different real roots as in Figure 9.

Case 4 When $t = 0$, the equation $y = t$ has different real roots as in Figure 10.

Case 5 When $t < 0$, the equation $y = t$ has no real solution as in Figure 11.

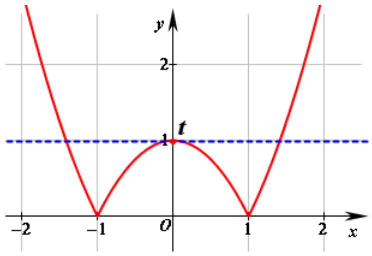

Figure 8

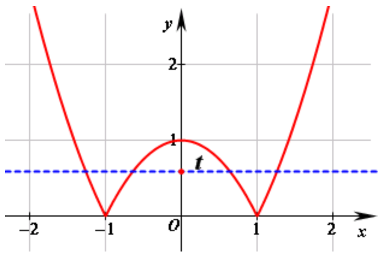

Figure 9

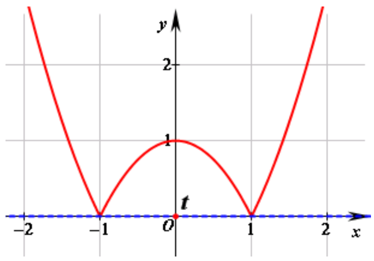

Figure 10

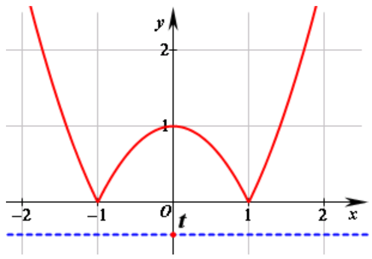

Figure 11

Next we consider the equation $h = x^2 - x + k$ with the axis of symmetry $x = {}^{1}/_{2}$ and the discriminant $\Delta = 1 - 4k$. So we have

Case a When $\Delta < 0$, as in Figure 12, the equation $h = x^2 - x + k$ does not have any real root. Therefore the corresponding equation $(x^2 - 1)^2 - |x^2 - 1| + k = 0$ does not have a real root either.

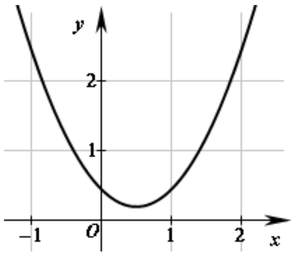

Figure 12

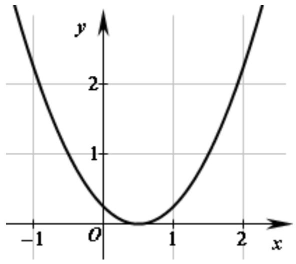

Figure 13

Case b When $\Delta = 0$, as in Figure 13, the equation $h = x^2 - x + k$ has two identical real roots $t_1 = t_2 = {}^1\!/_2$. Then the corresponding equation $(x^2 - 1)^2 - |x^2 - 1| + k = 0$ has four real roots as demonstrated in case 3.

Case c When $\Delta > 0$, the equation $h = x^2 - x + k$ has two different real roots t_1 and t_2:

Case c.1 If $0 < t_1 < t_2 < 1$, as in Figure 14, the equation $(x^2 - 1)^2 - |x^2 - 1| + k = 0$ has eight real roots as explained in case 3.

Case c.2 If $0 = t_1 < t_2 = 1$, as in Figure 15, the equation $(x^2 - 1)^2 - |x^2 - 1| + k = 0$ has five real roots as explained in case 2 and case 4.

Case c.3 If $t_1 < 0 < 1 < t_2$, as in Figure 16, the equation $(x^2 - 1)^2 - |x^2 - 1| + k = 0$ has two real roots as indicated in case 1 and case 5.

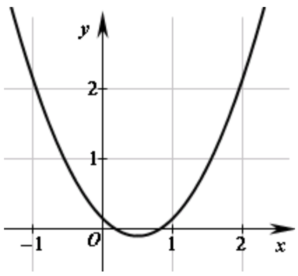

Figure 14

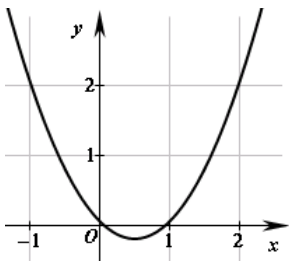

Figure 15

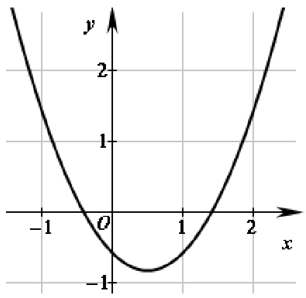

Figure 16

So the correct answer is “A”.

The following figures show that when k changes, the number of roots of the function curve $y = \left(x^2 - 1\right)^2 - \left|x^2 - 1\right| + k = 0$ also varies.

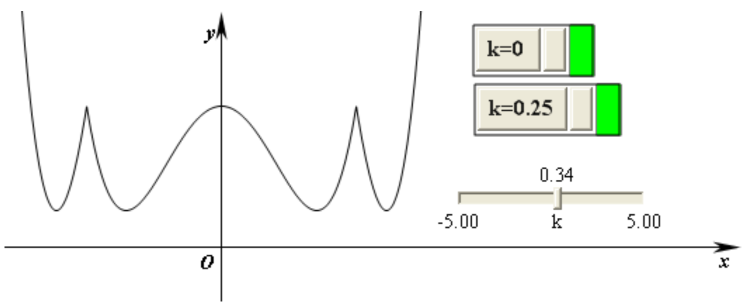

Figure 17

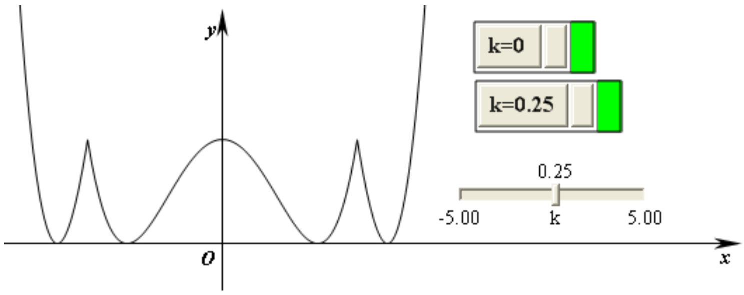

Figure 18

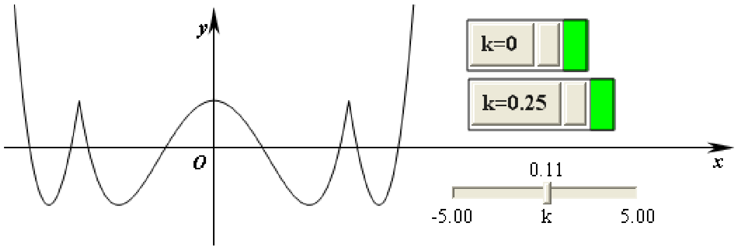

Figure 19

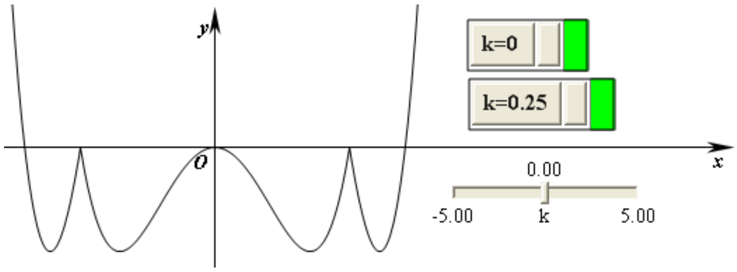

Figure 20

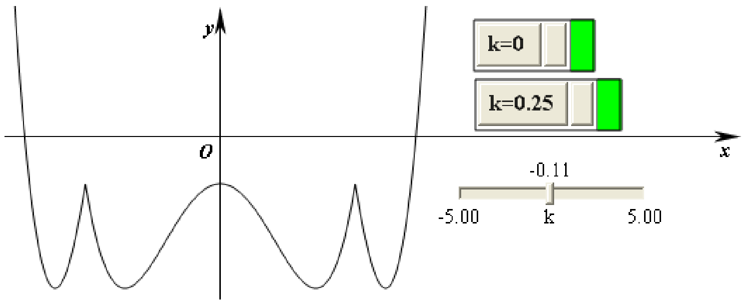

Figure 21