PROBLEM CORNER

Solutions to Problems from October 2012

Math Games

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Computer games may not only be entertainment, but also assist in the development of logic and analytical thinking. **Math games** help people learn mathematics in a fascinating way, and more deeply understand various areas such as Mathematical Statistics, Finance, Game Theory, Probability Theory, etc.

The following problems are created in form of games. The first problem doesn't require specific knowledge of mathematics. All we need is logical reasoning capabilities. The next two problems are **economic games**. We may just play and try to win the games intuitively. However, in problems 2 and 3 which are both economic games, we can obtain a so–called **optimal strategy** to maximize the player's benefit on average. The **optimal strategy** may be found by treating the game as solving a problem. To find correct solutions we need to be familiar with **Game Theory** and **Probability Theory**, understand concepts such as a **probability** of an event and its measurement, mathematical **expectation**, the mean value of a random variable, payoff calculation, optimal strategy, etc.

Problem 1

In Koschei's kingdom, there are four water sources numbered 1, 2, 3, and 4, whose taste and appearance are not different from the popular Russian Vodka. But Koschei's spring water has the following feature: it acts as a poison and a person has to take antidote to avoid death in 5 minutes after drinking a glass of Koschei's spring water. Koschei's water can be drinked only in whole glasses. If one drank two or more glasses of Koschei's water successively from a given spring, or one glass of water and one or more glasses of Vodka, the result is the same as he only drank one glass of Koschei's water. Suppose one drank one glass of Koschei's water from either springs 1, 2, or, 3. To obtain an antidote, one would then have to drink a glass of Koschei's water numbered higher than the first glass. If one drinks a glass numbered lower than the first glass, then the person would die instantly. Everyone has access to water from the springs numbered 1, 2 and 3. However, only Kischei can take water from the spring 4. Therefore, only Kochei can obtain an antidote when drinking a glass of water from spring 3.

Gamer Ivan challenged Koschei to a duel. Under their agreement, Ivan and Koschei sit in their shelters before the duel. They have access to water (except water from spring 4 which is for Koschei only). During the duel, they come out of the shelters, exchange a glass of liquid, drink it immediately and then come back to the shelters.

Find the optimal strategy for Ivan. Note that spies of Koschei can pry into Ivan's actions! Moreover, with the help of a time machine Koschei can predict the upcoming event 10 minutes ahead of time.

This game can be downloaded from: **http://deoma–cmd.ru/en/Products/MathGames.aspx.**

Please play and try to find a way to make Ivan alive. Too hard? Read the solution.

Solution

Assume that Ivan came to the duel without any preparation. Koschei may give him water from the spring 4. Don’t deceive ourselves; we have not the ghost of a chance. Being able to predict the upcoming result in next ten minutes, Koschei would not allow an outcome in favor of Ivan.

It is also reasonable to assume that Ivan could drink a glass of Koschei's spring water before he came to the duel. But remember that Koschei’s spies can pry into Ivan's actions! If Ivan drank spring water 2 or 3 in advance, Koschei would give him a glass of water from the spring 1. The water source of the second glass has the number smaller than the water source of the first glass, so Ivan dies instantly!

Now let us consider the following cases. In each case Ivan had drunk water from spring 1 before the duel. Then there are only two possibilities in the duel:

Case 1 Koschei gave water from the spring 1 or Vodka to Ivan.

Remember that if one drank two glasses of water successively from a given spring, the result is the same as he drank this water only once. So we may treat this case just as if Ivan drank the spring 1 water once.

Then the solution is easy: after coming back to the shelter, Ivan must drink the antidote from springs 2 or 3. Then Koschei's water has no effect on Ivan.

Case 2 Koschei gave spring water 2, 3 or 4 to Ivan. Then it is even better because Ivan got the antipoison for spring 1 water immediately and he didn’t need to drink anything else after returning to the shelter. Now again, Koschei's water drunk has no effect on Ivan.

These two cases are different but Ivan survives successfully in both of them. It seems that Ivan is destined to win. But it is too early to rejoice! Remember that Ivan could not know for sure what water he had to drink from Koschei in the course of the duel. So he still need to take the correct action after he came back to the shelter to stay alive.

What should Ivan do when he returned to his shelter after the duel? In the two cases described above, Ivan took different actions. But do you think it is really necessary to distinguish these two cases?

Let's try to combine the two cases into one. Now assume that Ivan always drank a glass of spring 1 water before he went to the duel.

Case 1 Koschei gave a glass of spring 1 water or vodka to Ivan in the duel. Ivan took another glass of water 1 after he returned to the shelter. Note that it is still equivalent for Ivan to drink the spring 1 water once. Then Ivan must drink a glass of water from spring 2 or 3 to detoxify his body.

Case 2 Koschei gave a glass of water from springs 2, 3 or 4 to Ivan. So Ivan got the antidote for spring 1 water that was taken previously. Again Ivan returned to the shelter and drank a glass of water from spring 1. It is dangerous for him, so he must immediately drink the antidote water from the springs 2 or 3.

Now two cases are unified for Ivan to win! Without knowing what water Koschei would provide at the duel, Ivan could remain alive whatsoever!

Let's summarize the optimal strategy for Ivan as in following:

The optimal strategy for Ivan:

Ivan drank a glass of water from spring 1 before the duel.
In the duel Ivan drank Koschei's glass of liquid.
After returning to the shelter, Ivan drank another glass of water from the spring 1.
Ivan immediately drank one glass of water from the springs 2 or 3.

In February 2013 children from Lyceum 1, Murmansk, Russia have found new solution, different from this one. Can you find it?

Economic Games

Problem 2

The gamer has a pair of dice. He can throw them no more than N times. After throwing the dice n times with $n < N$, the gamer can choose to take the sum of two numbers of the dice or continue playing. If the gamer has thrown the dice N times, he has to take the sum of the dice numbers.

Find the expected value of the payoff for the game depending on N. What is N for which the gamer should not take 10? How about the N value for which the gamer should not take 11?

You can install InMA 11 from:	**http://deoma–cmd.ru/en/Products/Algebra/InMA–11.aspx.**
To play this game, go to:	**6. Mathematics in economic games 6.1.7. GameDiceEnglish**

Solution

We use the Bellman principle (http://en.wikipedia.org/wiki/Bellman_equation) to solve multistage game. Each step of the multistage game results in creating a subgame. Each step requires a player to perform an action (to choose a strategy of behavior) so that the value of the subgame (an expected payoff for the rest game) has the highest value. To solve the problem, we construct a Bellman function containing the subgame values V_i, where V is the value of the subgame, i is the number of the moves taken by the players during the subgame. We use Vi to denote the value of the game with i rolls. Therefore, V_N is the value of a game with N rolls.

We will only give the solution for the game with one die. Please, find the solution to the pair of dice by yourself.

We find the values V_i for $i < N$ by solving the problem "backwards" (following in the opposite direction during the game).

The player has only one option for the last step of the game, that is to take the number which appears on the die. The probability of getting any one of six values from 1 to 6 is 1/6. So

$$V_0 = \frac{1+2+3+4+5+6}{6} = 3.5$$

The player has two choices for the penultimate step (and any other step) of the game: to take the number or continue playing. Note that at the i–th roll, it is advisable to take any number from rolling the die which is greater than the expected value Vi. Indeed, the expectation of winning a game given by Bayes' formula is

$$V_{i+1} = 6 \cdot \frac{1}{6} + max(5, V_i) \cdot \frac{1}{6} + max(4, V_i) \cdot \frac{1}{6} + V_i \cdot \frac{1}{2}.$$

$$V_{i+1} = 1 + \frac{5}{6} V_i, V_i \geq 5;$$

$$V_{i+1}=\frac{11}{6}+\frac{2}{3}V_i,\, 5>V_i\geq 4;$$

$$V_{i+1}=\frac{5+V_i}{2},4>V_i.$$

For the game ends after one throw, so the gamer get 4, 5 or 6 (each of them is greater than V_0 = 3.5). Then the expected value of subgame 1 is

$$V_1=\frac{5+V_0}{2}=\frac{17}{4}=4.25.$$

For the game ends after two throws, so one gets only 5 and 6 (each of which exceeds V_1 = 4.25). Then the expected value of subgame 2 is

$$V_2=\frac{11}{6}+\frac{2}{3}\times\frac{17}{4}\approx 4.67$$

If the game has three throws, so one gets only 5 and 6 (each of which exceeds V_2 = 4.67). Then the expected value of subgame 3 is

$$V_3=\frac{11}{6}+\frac{2}{3}\times\frac{14}{3}\approx 4.94$$

Similarly for a game with four throws, so the gamer gets 5 and 6 (each of which exceeds V_3 = 4.94). The expected value of subgame 4 is

$$V_4=\frac{11}{6}+\frac{2}{3}\times\frac{89}{18}\approx 5.13$$

For a game ends with five or more throws, so the gamer only gets 6 > Vi =5.13. Then expected value of the subgame i+1 with $i \geq 4$ is

$$V_{i+1}=1+\frac{5}{6}V_i.$$

Now let’s summarize the optimal strategy for the one die game as in follows:

The optimal strategies for one die game.

Let the game be ended after i throws.
If $i \geq 5$, then the gamer should not take any number less than 6.
If i = 2, 3, or 4, then the gamer should not take any number less than 5.
If i = 1, then the gamer should not take any numberless than 4.

Please, find the solution to the pair of dice by yourself.

Answer

Let the game be ended after i throws.
If i ≥ 9, then the gamer should not take any number less than 11.
If i ≥ 25, then the gamer should not take any number less than 12.

Problem 3

A sequence contains N random distinct numbers and the gamer can only see one number at a time. At a number provided by the game, the gamer can choose to decline it or accept it. The game is over whenever the gamer accepted a number. If the gamer has declined the first N-1 numbers, he has to accept the last number. The gamer wins if he selected the biggest number of the sequence.

Suppose that each position of the sequence has an equal chance to have the biggest number, find the value of the game (probability of winning) and the optimal strategy for the game as a function of N.

You can install InMA 11 from: **http://deoma–cmd.ru/en/Products/Algebra/InMA–11.aspx.**
To play this game, go to: **6. Mathematics in economic games 6.1.8. gameNumberEnglish**

Solution

Suppose we have a sequence consisting of N random distinct numbers. From now on, without confusion, we use N to denote the biggest number, 1 as the smallest number, and the other numbers as follows:

$$N > N - 1 > \ldots > 3 > 2 > 1.$$

For example, suppose three random numbers 3, 6, and 9 are given in a sequence. Then we have $N = 3$. We denote the smallest number 3 as "*1*", the largest number 9 as "*3*", and the middle number 6 as "*2*". If the sequence listed the three numbers in order "6, 3, 9", then for this case we use the notation "*213*". In the case "*213*", the gamer wins if he skips the numbers *2* and *1*, and chooses the last number *3*.

In general, if we accepted a number in the sequence randomly, then the chance of winning is *1/ N*. For instance, *1000* numbers are given in a sequence. The value of the game, or the probability of winning, is equal to *1/1000*. However, in the following we will show a game strategy under which the probability of winning the game is approximately *1/3*.

At first, we explore the simplest cases for $N = 2, 3, 4, 5$. Remember that in the following examples, N denotes the largest number, *1* denotes the smallest number of the sequence, etc.

Biggest number for a sequence with N=2 numbers

The numbers 1 and 2 may be given to the gamer in two orders: "12" or "21". The gamer may choose the following strategies:

(1) he always accepted the first number; Or
(2) he always accepted the second number.

The value of the game is ½ for either strategy.

Biggest number for a sequence with N=3 numbers

The numbers 1, 2 and 3 may be presented in six different orders 123, 132, 231, 213, 312 and 321. The gamer may

(1) always choose the first number in the sequence. In this case, the probability of winning is 1/3; Or
(2) always decline the first number and choose the first number from the rest which is bigger than the skipped number. The gamer wins in the cases of 132, 231 and 213. So the probability of winning is 1/2.

Biggest number for a sequence with N=4 numbers

The numbers 1, 2, 3 and 4 may be presented in 4!=24 ways:

4123, 4132, 4231, 4213, 4312, 4321, 1423, 1432, 2431, 2413, 3412, 3421;
1243, 1342, 2341, 2143, 3142, 3241, 1234, 1324, 2314, 2134, 3124, 3214.

Possible strategies for the gamer are:

(1) he always chooses the first number in the sequence. In this case, the probability of winning is 1/4.

(2) he always skips the first number and then chooses the first one from the numbers coming later, which is greater than the skipped number. He wins in 11 cases: 1423, 1432, 2431, 2413, 3412, 3421, 2143, 3142, 3241; 3124, 3214. The probability of winning is 11/24.
(3) he always skips the first two numbers and then chooses the first number from the rest in the sequence which exceeds the skipped two values. He wins in 10 cases: 1243, 1342, 2341, 2143, 3142, 3241; 1324, 2314, 3124, 3214. Then the probability of winning is 10/24.

Biggest number for a sequence with N=5 numbers

The numbers may be listed in 5!=120 different orders. Suppose we skipped the first ***k*** numbers where $1 \le k < N$ and chose the first number from the rest numbers which is greater than the maximum of the skipped numbers.
If the greatest number of the sequence is in position 1, ..., ***k*** we lose; If the greatest number of the sequence is in position (***k*** + 1), we won. The probability for this case is $1/N = \mathbf{1/5}$.
Now suppose the greatest number is on position ***n*** with $k+1 < n \le N$, then probability of this event only is $1/N = \mathbf{1/5}$. To study the winning probability in this case, let's firstly look at two special examples 12345 and 34125 for k = 2. In case 12345, we lose because 3 > 2. In case 34125, we win because we skip 1<4 and 2<4 and get 5. Then the player wins if the greatest number in positions 1, 2, …, $n-1$, the positions before ***n*** where the maxima value ***N*** locates, lies among the firstly skipped ***k*** numbers. For example, in the winning case 34125, the biggest number before position 5 is 4 which is in position 2 and will be declined in the k-number-skip-strategy. Therefore, for a given ***n*** value, the probability of putting the maximum value of positions 1, 2, …, $n-1$ into one of the first k positions is $\frac{k}{n-1}$.
Hence, for the k-number-skip-strategy, and suppose the largest number of the sequence locates on a position>k+1, then the probability of winning is

$$p(k) = \frac{1}{5}\sum_{n=k+1}^{5}\frac{k}{n-1} = \frac{k}{5}\sum_{n=k+1}^{5}\frac{1}{n-1}$$

If $k = 1$, then

$$p(1) = \frac{1}{5}\sum_{n=2}^{5}\frac{1}{n-1} = \frac{1}{5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) = \frac{5}{12}$$

If $k = 2$, then

$$p(2) = \frac{1}{5}\sum_{n=3}^{5}\frac{1}{n-1} = \frac{2}{5}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) = \frac{13}{30} > \frac{5}{12}.$$

If $k = 3$, then

$$p(3) = \frac{3}{5}\sum_{n=4}^{5}\frac{1}{n-1} = \frac{3}{5}\left(\frac{1}{3}+\frac{1}{4}\right) = \frac{7}{20} < \frac{13}{30}.$$

Note that the greatest probability occurs when $k = 2$, and 13/30 >1/5. So we summarize the optimal strategy is the following:
We ignore the first $k = 2$ numbers and take the first offered number which is greater than the maximum of the skipped numbers. And the game value is 13/30.

Biggest number for a sequence with N numbers

Let us use the following strategy: we skip the first ***k*** numbers and choose the first number from the rest which exceeds the maximum of the skipped numbers. If the greatest number is in position ($k + 1$), the player wins with probability $1/N$.
Let the greatest number be in position ***n*** with $n > k + 1$. **So** the probability of this event is $1/N$.
Then the player wins if the greatest number of positions 1, 2, …, $n-1$, is in the first ***k*** positions. So the probability of this event itself is $\frac{k}{n-1}$.

Hence, for the k-number-skip-strategy, and suppose the largest number of the sequence locates on a position>k+1, then the probability of winning is

$$p(k) = \frac{1}{N}\sum_{n=k+1}^{N}\frac{k}{n-1} = \frac{k}{N}\sum_{n=k+1}^{N}\frac{1}{n-1}$$

The probability increases from ***1/N*** for ***k*** **=** 0, reaches the maximum and then starts to decrease again to the same value of ***1/N*** for ***k*** **=** ***N*** – 1. To find ***k*** value which corresponds to the highest probability of winning, we compare the successive probabilities:

$$p(k) > p(k+1) \Leftrightarrow \frac{k}{N}\sum_{n=k+1}^{N}\frac{1}{n-1} > \frac{k+1}{N}\sum_{n=k+2}^{N}\frac{1}{n-1} \Leftrightarrow 1 > \sum_{n=k+2}^{N}\frac{1}{n-1}$$

$$p(k) > p(k-1) \Leftrightarrow \frac{k}{N}\sum_{n=k+1}^{N}\frac{1}{n-1} > \frac{k-1}{N}\sum_{n=k}^{N}\frac{1}{n-1} \Leftrightarrow \sum_{n=k+1}^{N}\frac{1}{n-1} > 1$$

We use the summation rule:

$$\sum_{n=k}^{N} f(n) \approx \int_{k-0.5}^{N+0.5} f(x)dx$$

We obtain

$$\sum_{n=k+2}^{N}\frac{1}{n-1} \approx \int_{k+1.5}^{N+0.5}\frac{1}{x-1}dx = \ln(x-1)\Big|_{k+1.5}^{N+0.5} = \ln\frac{N-0.5}{k+0.5} < 1, \text{ where } k > \frac{\mathrm{N}-0.5}{\mathrm{e}} - \frac{1}{2}$$

$$\sum_{n=k+1}^{N}\frac{1}{n-1} \approx \int_{k+0.5}^{N+0.5}\frac{1}{x-1}dx = \ln(x-1)\Big|_{k+0.5}^{N+0.5} = \ln\frac{N-0.5}{k+0.5} > 1, \text{ where } k < \frac{\mathrm{N}-0.5}{\mathrm{e}} + \frac{1}{2}$$

Hence, the optimal ***k*** value is

$$k_0 = \left[\frac{N-0{,}5}{e}\right]$$

Answer

The value of the game – the probability of winning with the optimal ***k*** is

$$p(k) = \frac{k_0}{N}\sum_{n=k_0+1}^{N}\frac{1}{n-1} \approx \frac{k_0}{N}\ln\frac{2N-1}{2k_0-1}$$

This approximate formula for ***N*** > 4 leads to an error less than 1%. For ***N*** = 3, the value of game is 0.536 instead of 0.5. For ***N*** = 4 the value of game is 0.486 instead of 0.458. For *N*> *10* the probability of winning is close to 0.38.

The optimal strategy

The optimal strategy of the game is to skip the first $k_0 = \left[\frac{N-0.5}{e}\right]$ numbers and take the first number of the rest which is larger than the maximum of the skipped numbers.