# PROBLEM CORNER

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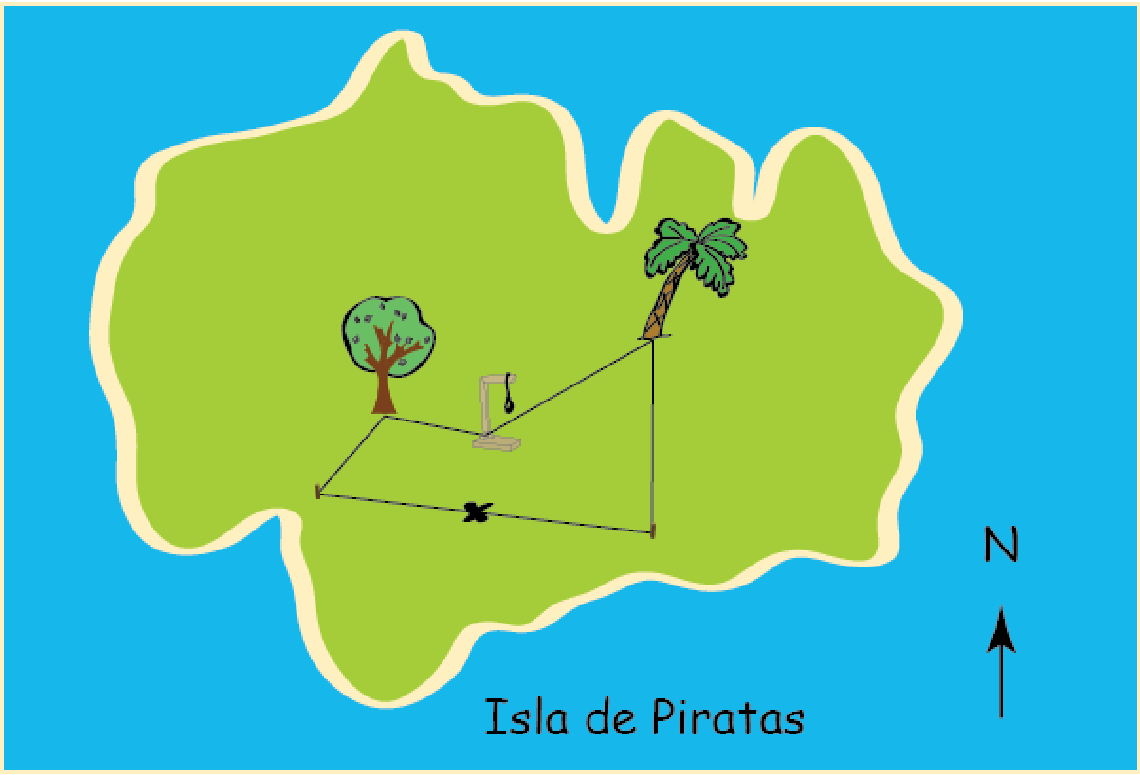
## Problem 1

Esteban was going through the attic of his grandfather's house and found a paper describing the location of a buried treasure on a particular Island. The note said that on the island one would find a gallows, an oak tree, and a palm tree. To locate the treasure one would begin at the gallows, walk to the palm tree, turn right 120˚ and walk the same number of paces away from the palm tree. Drive a spike into the ground at this point. Then return to the gallows, walk to the oak tree and turn left 60˚ and walk the same number of paces away from the oak tree. Drive a second spike in the ground at this point. The midpoint of a string drawn between the two spikes would locate the treasure. Esteban and his friends mounted an expedition to the island, found the oak tree and the palm tree mentioned in the note, but the gallows had rotted away long ago. They returned home with the map above and no treasure.

Can you help these young men find the location of the treasure?




Reference: by http://jwilson.coe.uga.edu/emt725/Treasure/Treasure.html

## Solution

To better describe the problem mathematically, let's impose a simple coordinate system.. Assume that the Oak is located at $(0,0)$ and that the Palm is located at $(d, 0)$. So the line uniquely determined by the Oak tree and Palm tree is the $x$-axis. Then the general picture is shown in Figure 2 below with Gallows at $(a, b)$. First let's consider an easier case of this problem, that the gallows is located half-way between the oak and the palm at $(d/2,0)$. Then angle $x = 120°$ and angle $y = 60°$, producing two $30 - 60 - 90$ triangles with hypotenuse length $d/2$.

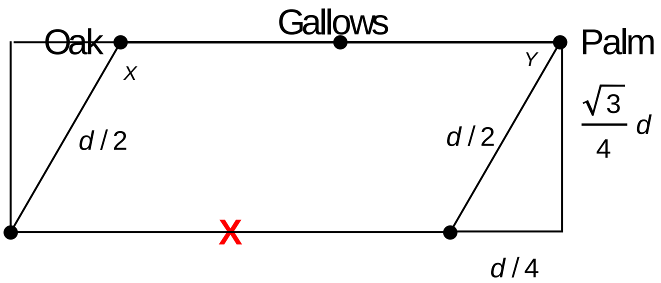



Figure 1

So the midpoint is located at $x = (-d/4 + (d - d/4))/2 = d/4$ and $y = -(\sqrt{3}/4)d$. So to find the treasure, start at the Oak tree, walk 1/4 of the distance toward the Palm tree. Turn right and walk $\sqrt{3}$ times the distance just covered. This is $\sqrt{3}\, d/4 \;=\; \sqrt{3}/4\, d$, as shown in the Figure 1.

Now we will consider the general case that the Gallows locate at $(a, b)$ as shown in Figure 2. With some help from compute algebra we can show that the treasure always locates at point $(d/4, -(\sqrt{3}/4)d)$, which is indeed independent of the gallows location.

Define the Oak at (0,0), the Palm at $(d, 0)$ and the Gallows in general position $(a, b)$. So

$$\boldsymbol{oak = (0,0), palm = (d,0), gallows = (a,b), m1 = -2\pi/3; m2 = \pi/3;}$$

A vector form the gallows to palm is given as

$$\boldsymbol{v1 = palm - gallows =< d - a, -b >^T}$$

From Linear Algebra we know that a rotation for 120° right turn at the palm can be represented as a matrix

$$m1 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

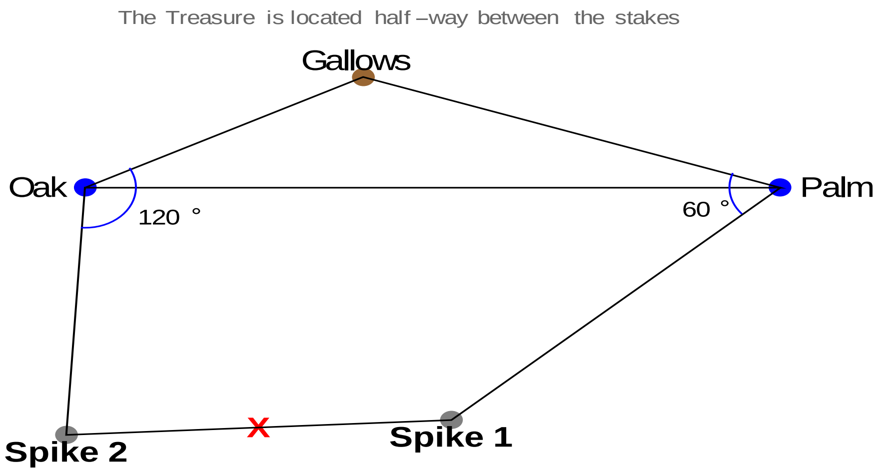



Figure 2

Compute the direction vector $\boldsymbol{v}3$ after the turn at the palm tree (note that the length is preserved by the rotation):

$$v3 = m1 \cdot v1 =< 1/2\ (a - \sqrt{3}b - d), 1/2\ (\sqrt{3}\ (a - d) + b) >^T$$

The vector form for the position of the first stake is the addition of the palm vector and $\boldsymbol{v}3$:

$$s1 = palm + v3 =< 1/2\ (a - \sqrt{3}b + d), 1/2\ (\sqrt{3}\ (a - d) + b) >^T$$

A vector from gallows to oak is given as:

$$\boldsymbol{v2} = \boldsymbol{palm} - \boldsymbol{gallows} =< -\boldsymbol{a}, -\boldsymbol{b} >^{\boldsymbol{T}}$$

And From Linear Algebra we know that a rotation for 60° left turn at the Oak can be represented as a matrix

$$m2 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Compute the direction vector $\boldsymbol{v}4$ after the left turn 60° at the Oak tree as:

$$\boldsymbol{v}4 = m2 \cdot \boldsymbol{v}2 =< 1/2\ (\sqrt{3}b - a), 1/2\ (-\sqrt{3}\ a - b) >^T$$

Then similarly we can determine the location of the second stake as the oak plus $\boldsymbol{v}4$:

$$s2 = Oak + \boldsymbol{v}4 =< 1/2\ (\sqrt{3}b - a), 1/2\ (-\sqrt{3}\ a - b) >^T$$

Now compute the location of the midpoint between the two stakes:

$$t = \frac{s1 + s2}{2} =< \frac{d}{4}, \frac{\sqrt{3}d}{4} >$$

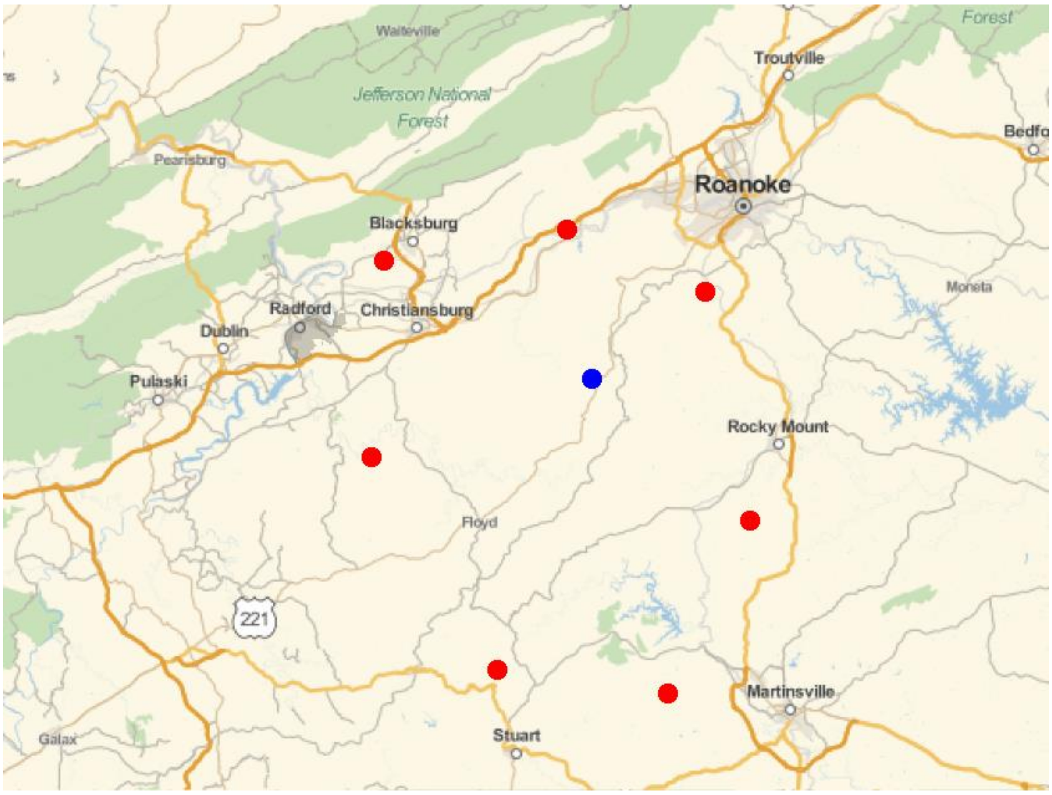
We can see that this answer is exactly the same as in the case that the gallows locates half-way between the Oak and the Palm tree at $(d/2,0)$.

Interested readers can find the Mathematica codes for the location of treasure for any input $a$, $b$, and $d$ values here. Furthermore, a manipulative version of the problem solution in Computable Document Format (CDF), in which interested readers can visualize the result that the treasure location is independent of the Gallows location, can be found here.

## Problem 2

A set of points $(x_i, y_i)$ is given in the table below, representing the geographic location of a set of airfields that can be used for emergency landing. These are shown as red points on the map below. A new airfield is added at $P = (8,5)$, (not included in the table below), shown as a blue dot on the map. Define the region **R** as the set of points that are closer to the point $P$ than to any of the points in the table. Then an airplane that encounters an emergency while over any point in this region should attempt to land at $P$. Determine the exact area of the region **R**. Assume that the coordinates are in kilometers from a conveniently chosen origin and assume the points lie in a plane. Use the standard distance formula in the Cartesian plane to measure the distance between the points.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | 27 | 19 | -10 | -32 | -29 | -2 | 23 |
| $y$ | 1 | 23 | 30 | 17 | -10 | -30 | -28 |



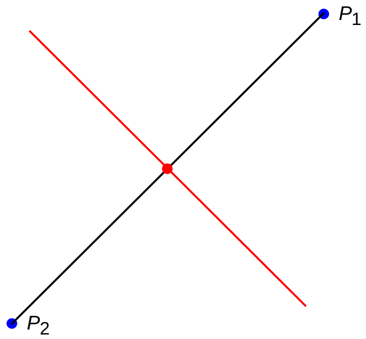

Bonus question: What effect would using great circle distance have on the shape and area of this region assuming the coordinates given are (latitude, longitude) in degrees? Note, these numbers as degrees would mean the points cover a much larger geometric region than that shown in the map above.

## Solution

Let's first of all consider a simplified case: Given two points $P1$ and $P2$ on a plane, we want to find the region such that every point in that region is closer to point $P1$ than to point $P2$. Assume that $P1 = (0,0)$, $P2 = (a, b)$, and assume that

$$< x, y > = < \frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} >$$

is the unit vector in the same direction of vector $\overrightarrow{P1P2}$. Then $< y, -x >$ is the unit vector in the direction of the perpendicular bisector of line segment $P1P2$. Notice that the perpendicular bisector divides the plane into two regions: one with all points that are closer to $P1$ and the other region has all points closer to point $P2$. In the figure below, all points above the red line are closer to $P1$ than to $P2$.




Now suppose there are a set of point $P1 = (x_1, y_1)$, $P2 = (x_2, y_2)$, … … , $Pn = (x_n, y_n)$. For any pair of points $Pi$ and $Pj$, we can construct the perpendicular bisector of $PiPj$. After creating the set of all perpendicular bisectors between the pair of points, we will determine all of the intersection points between all distinct pairs of these lines because the intersection points are potential corners of the region that is closer to the new airfield location P (the blue point in Figure 3) than to any other airfield.

In Figure 3 below, the green points represent the existing airfield locations, the blue point is the new airfield. The blue lines are perpendicular bisectors between pairs of points, and the red points are points of intersection between the perpendicular bisectors. Visual inspection of this plot reveals the polygon that bounds the region closest to the new airport. Using software Mathematica, we can calculate these intersection points of the perpendicular bisectors, in counter-clockwise order in Table 1.

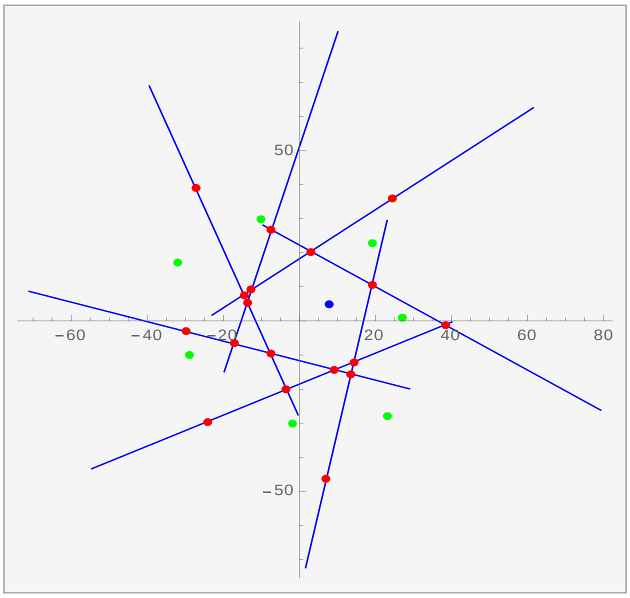



Figure 3

| $x$ | $\frac{3627}{1198}$ | $\frac{7371}{386}$ | $\frac{5419}{378}$ | $\frac{1063}{114}$ | $-\frac{3519}{458}$ | $-\frac{397}{29}$ | $-\frac{4917}{392}$ | $\frac{3627}{1198}$ |
|---|---|---|---|---|---|---|---|---|
| $y$ | $\frac{24439}{1198}$ | $\frac{2042}{193}$ | $-\frac{4547}{378}$ | $-\frac{1631}{114}$ | $-\frac{4327}{458}$ | $\frac{467}{87}$ | $\frac{1801}{196}$ | $\frac{24439}{1198}$ |

Table 1: Intersection points of the perpendicular bisectors in Figure 3, in counter-clockwise order

It is well known that given a polygon with vertices $(x_1, y_1), (x_2, y_3), \dots (x_n, y_n)$, the area of polygon can be calculated using the following formula:

$$Area = \frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \cdots \begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix}\right) \qquad (1)$$

Interested readers can find more explanation on using formula (1) to find the area of a polygon from [1]. Therefore we use the coordinates listed in Table 1 into equation (1), the area of the polygon as shown in Figure 3 is:

$$Area = {}^{476327781117439999}/_{617562668445408} \approx 771.303\ unit^2$$

A more formal approach is to construct a ***Voronio diagram*** on the point set. In mathematics, a Voronoi diagram is a partitioning of a plane into regions based on set of airfield locations, Figure 4 is a Voronoi diagram produced by the Mathematica command VoronoiMesh: it divides the plane into a collection of regions closest to each of the points. For instance, in Figure 4, Region $\mathcal{A}$ contains all points closer to point `P` than any other points in the plane; Region $\mathcal{B}$ represents the collection of all points closer to point `P1` than any other points in the plane, etc. Interested readers can learn more about the Voronoi diagram from references [2].

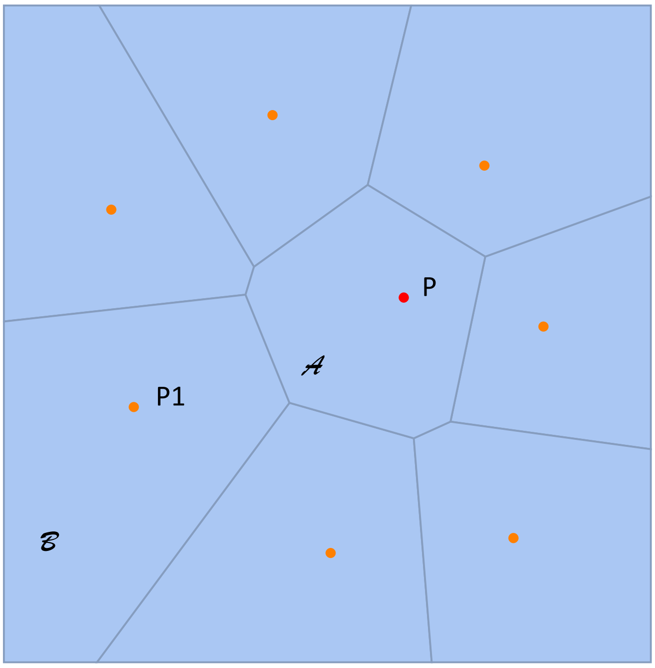



Figure 4

Interested readers can find the Mathematica code for drawing perpendicular bisectors, calculating polygon areas, drawing Voronoi diagrams for a set of points here.

## Reference

[1] "*Polygon Area from Wolfram Mathworld* ",
http://mathworld.wolfram.com/PolygonArea.html

[2] "*Voronoi Diagram from Wolfram Mathworld* ",
http://mathworld.wolfram.com/VoronoiDiagram.html