The Problem Corner (solutions)

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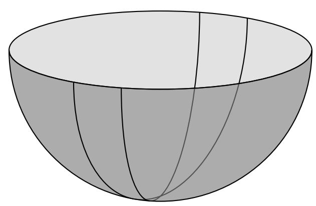
1. Let us consider a simple optimization problem. Given a function of two variables $z = f(x, y)$, we can search for its extremal points by intersecting its graph with *vertical* planes, and looking at the resulting curve. Suppose that for each vertical plane passing through the origin, the intersection with the graph of the function is a curve having a minimum at the origin. Is it true that $f(x, y)$ will have a minimum at the origin? Prove it if you think it is true, or give a counterexample otherwise.



Solution: In general, it is false that the function $z = f(x, y)$ will have a minimum. As a counterexample, one can consider the function

$$f(x, y) = (y - x^2)(y - 2x^2)\,. \tag{1}$$

For any line $y = ax$ with $a \neq 0$, we get the corresponding intersection $z = 2x^4 - 3ax^3 + a^2x^2$, which has a minimum in $x = 0$ (that in turn determines the origin, as $y = ax = 0$ too), as it is easy to check by taking derivatives.
In the particular case of the section given by $x = 0$ we get the parabola $z = y^2$, which also has a minimum at the origin.
However, it is clear from (1) that for those non-zero (x, y) such that $y = ax^2$ with $1 < a < 2$, it is $z < 0$, and hence $f(x, y)$ can not have a minimum at the origin.

2. A certain sport is played in two halves, and there is the figure of a penalty: a free shot as a consequence of a fault. In analyzing the performance in penalties, the sport section of a newspaper mentions that Team A had a better performance in both halves, so it was overall better. A dissenting reader, Mr. Simpson, writes complaining that the numbers really say that his team (B, of course), was better. Is this possible?

Solution: Yes, the two facts are compatible (and this is a well-known statistical phenomenon called the Simpson's paradox). Consider, for example, the following figures about performance in penalties:

Team	1st Half	2nd Half
A	5 out of 6	7 out of 19
B	11 out of 14	2 out of 6

It is clear that $5/6 > 11/14$, so team A was better in the first half. Also, as $7/19 > 2/6$ the team A was better in the second half. However, globally team A had an efficiency of $12/25$, while team B had $13/20$, with $13/20 > 12/25$.