PROBLEM CORNER

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Problem 1

Consider parallelogram ABCD, $|AB| \neq |BC|$. Let E be the intersection of the perpendicular to the diagonal AC dropped from the point D with the line BC and let F be the foot of the perpendicular from the point B to the line DE. Assuming that the lines CF and AE perpendicular, determine the angle ACB.

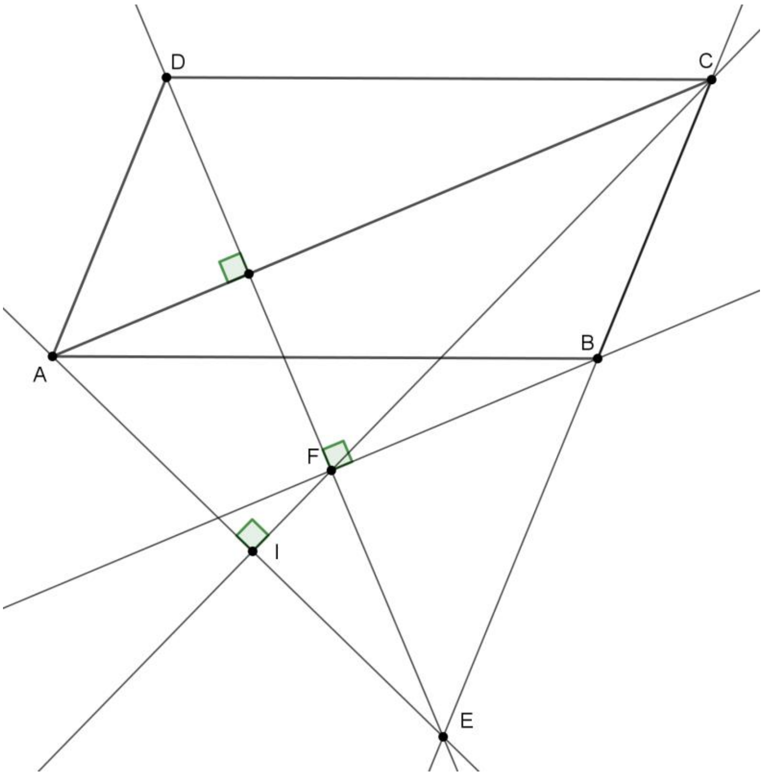

Figure 1 – Parallelogram

SOLUTION

The point F is the orthocenter of the triangle AEC, hence AF is perpendicular to BC. As the segments AD and BC are equal and parallel, the diagonal AC is a bisector of the segment DF and the triangle DAF is isosceles. From that follows

$$\sphericalangle ACB = \sphericalangle DAC = \sphericalangle CAF$$

Hence $\sphericalangle ACB = 45°$.

Problem 2

Consider a triangle ABC and its circumscribed circle k. On the circle choose an arbitrary point P and inside the triangle select an arbitrary point G. Consider circles GAB, GBC, GCA. Denoting $P_{AB},\ P_{BC},\ P_{CA}$ the inverse images of the P with respect to the circles, prove or answer following statements:

a) Points $P_{AB},\ P_{BC},\ P_{CA}$ and G lie on a circle C.
b) As P moves along the circle k, the centre of the circle C moves along a line.
c) Determine in the triangle a point $G = G_L$ in such a way that the points P_{AB}, $P_{BC},\ P_{CA}$ and G_L are always collinear (we consider a line as a special case of a circle).

Hint: Apply the Simson-Wallace theorem.

Solution

a) Let us start by the Simson-Wallace Theorem [1]:
Let ABC be a triangle and P a point in a plane. If P lies on the triangle's circumcircle, then its images in reflection with respect to the sides of the triangle ABC and the orthocentre of the triangle are collinear – they lie on the so-called Simson line.

Let consider inverse image of the geometrical objects from the task with respect to an arbitrary circle with the centre G. Because the relation "point – inverse circle – image" is preserved by inverse mapping, we can reformulate the problem exactly as a statement of S.-W. theorem: The three circles are mapped to lines, the points $P, P_{AB},\ P_{BC},\ P_{CA}$ are mapped to points $P', P'_{AB},\ P'_{BC},\ P'_{CA}$ and the points $, P'_{AB},\ P'_{BC},\ P'_{CA}$ are reflections of the point P' with respect to these lines. Because the point P lies on the circumscribed circle of the inverse image of the triangle ABC, the points $P'_{AB},\ P'_{BC},\ P'_{CA}$ lie on a Simson line. Further, the image of the point G lies in infinity and hence lies on the line too. As the inverse images of the four points lie on a line, the original points $P_{AB},\ P_{BC},\ P_{CA}$, G must lie on a circle C or on a line, which is a special case of circle.

b) The most straightforward way is to prove that the circle C passes through two fixed points. The first point is obviously G. From the S.-W. theorem, we know that the Simson line passes through the orthocentre H, hence its image – the circle C – passes through the inverse image of H. It is the second point.

c) We will again use the inverse images of the objects from the task: The Simson line passes through the orthocentre H. In order to preserve this line under mapping, it is necessary $G_L = H$, hence G_L must be the orthocentre of the image of the original triangle ABC. Consider points A, B and its inverse images A', B'. Let the centrum of an inverse circle be a point G. As is obvious from fig 2, the angles are swapped by the mapping (justification is left to the reader).

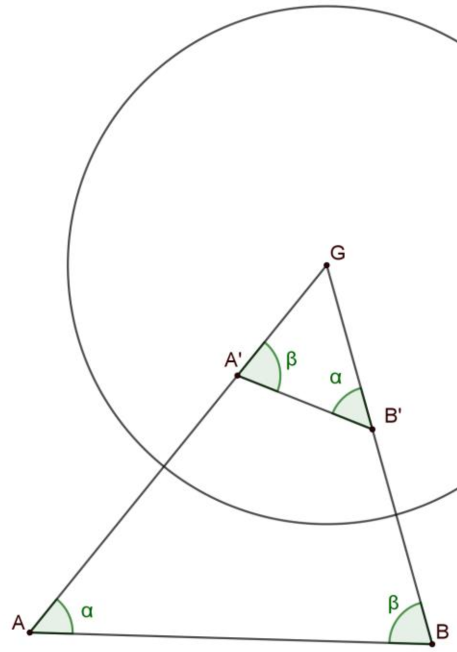

Figure 2 – inverse mapping

From that follows that the point G_L is the orthocentre of the inverse imagine of the triangle ABC if and only if it is the incentre or excentre of the triangle ABC. As the point G_L is inside the triangle, it is its incentre.

[1] - http://home.pf.jcu.cz/~sbml/wp-content/uploads/Skrisovsky_web.pdf

Problem 3

On a circle k are arbitrarily selected points A, B, C, D. Denote the orthocenter of the triangle ABC as H_D and analogically introduce the orthocenters H_A H_B H_C. Prove that the orthocenters lie on a circle with its radius equal to the radius of the circle k.

SOLUTION

We will solve the problem by means of vectors. Let the center of the k is O and let it be the origin of the coordinate system. It is easy to show that

$$H_D = \vec{A} + \vec{B} + \vec{C} = \left(\vec{A} + \vec{B} + \vec{C} + \vec{D}\right) - \vec{D}$$

and that the point H_D lies on the circle with the center $S = \left(\vec{A} + \vec{B} + \vec{C} + \vec{D}\right)$ and radius $R = \left|\vec{D}\right| = \left|\vec{A}\right| = \left|\vec{B}\right| = \left|\vec{C}\right|$.

For the other three points we arrive at the same conclusion.