PROBLEM CORNER

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Let *Q* be a convex quadrilateral with vertices A, B, C, D.

We call edges of *Q* the four sides and the two diagonals, AB, BC, CD, DA, AC, BD.

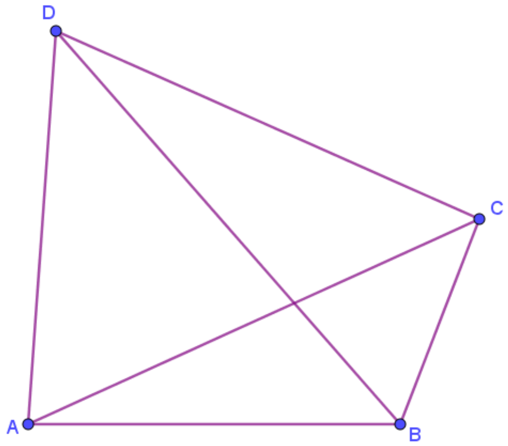

Figure 1. The quadrilateral *Q*

Problem 1

Let M_1, M_2, M_3, M_4, M_5, M_6 be the midpoints of the edges AB, BC, CD, DA, AC, BD.

Prove that the segments M_1M_3, M_2M_4, M_5M_6 are concurrent in a point G that bisects them all.

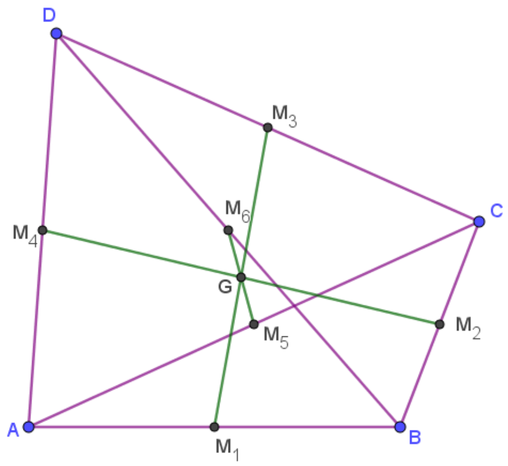

Figure 2. *Q* and the midpoint segments

SOLUTION

In the triangle ABC, the segment M_1M_2 joins the midpoints of the edges AB and BC, then M_1M_2 is parallel to AC and $M_1M_2 = \frac{1}{2}$ AC. Analogously, the segment M_3M_4 is parallel to

AC and $M_3M_4 = \frac{1}{2}$ AC. Therefore, $M_1M_2M_3M_4$ is a parallelogram. The common point G of its diagonals bisects both of them, M_1M_3 and M_2M_4.

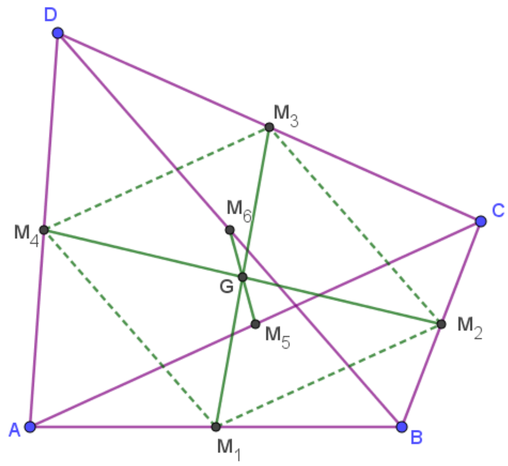

Figure 3. *Q* and the parallelogram $M_1M_2M_3M_4$

Let us consider different cases on *Q*.

<u>Case 1</u>. *Q* does not have any pairs of opposite parallel sides.

In the triangle ABC, the segment M_1M_5 joins the midpoints of the edges AB and AC, then M_1M_5 is parallel to BC and $M_1M_5 = \frac{1}{2}$ BC. Analogously, the segment M_3M_6 is parallel to BC and $M_3M_6 = \frac{1}{2}$ BC. Therefore, $M_1M_5M_3M_6$ is a parallelogram. Its diagonals bisects each other and since G is the midpoint of M_1M_3 then G is also the midpoint of M_5M_6.

Therefore, the segments M_1M_3, M_2M_4, M_5M_6 are concurrent in a point G that bisects them all.

Observe that also the quadrilateral $M_4M_5M_2M_6$ is a parallelogram.

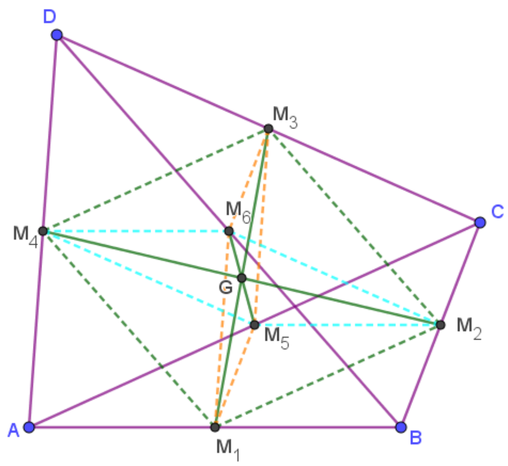

Figure 4. *Q* in case1

<u>Case 2</u>. *Q* has exactly one pair of opposite parallel sides.

Assume that AB is parallel to CD.

$M_4M_5M_2M_6$ does not exist anymore (because the segments M2M6 and M4M5 are parallel to BC and the segments M4M6 and M5M2 are parallel to AB. Since AB is parallel to CD they are all parallel to each other, therefore the points M_2, M_5, M_4 and M_6 are collinear and M_5M_6 is contained in M_2M_4), but the parallelograms $M_1M_2M_3M_4$ and $M_1M_5M_3M_6$ still hold and, since they share the diagonal M_1M_3 then they all meet in a point G that bisects M_1M_3, M_2M_4, M_5M_6.

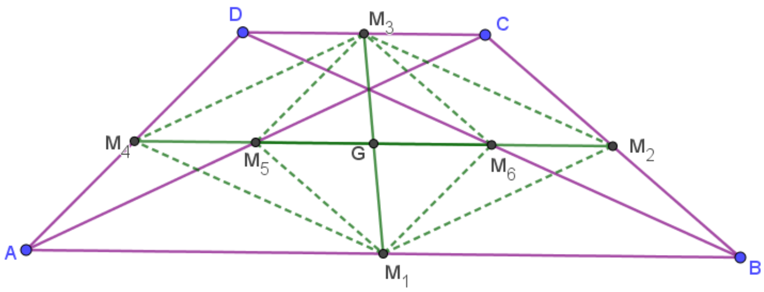

Figure 5. *Q* in case 2

Case 3. *Q* is a parallelogram.

If *Q* is a parallelogram then the parallelograms $M_4M_5M_2M_6$ and $M_1M_5M_3M_6$ do not exist anymore because M_5 and M_6 coincide with G (being G midpoint of the diagonals AC and BD).

Then the problem is solved also in this case.

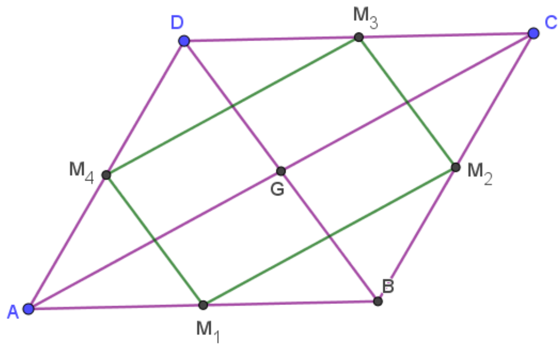

Figure 6. *Q* in case 3

Problem 2

Let A', B', C' and D' be the centroids of the triangles BCD, ACD, ABD and ABC respectively.

Prove that

- the segments AA', BB', CC' and DD' are concurrent in G;
- G divides each segment in two parts, the one containing the vertex twice the other one.

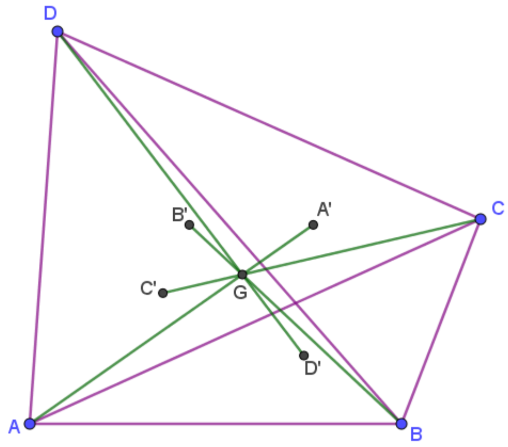

Figure 7. *Q* and centroid segments

SOLUTION

Let M_2 be the midpoint of BC. The segment DM_2 is a median of the triangle BCD, therefore it contains the centroid A'. Let N be the midpoint of DA' and M_4 the midpoint of AD. The segment NM_4 joins the midpoints of the edges DA' and DA of the triangle DAA', then NM_4 is parallel to AA' and AA'=2 NM_4.

Let G be the common point of AA' and M_2M_4. Let us prove that G is the midpoint of M_2M_4. In fact, the segment GA' is parallel to NM_4 and passes through the midpoint A' of the edge NM_2 of the triangle NM_2M_4, then G is the midpoint of M_2M_4. Moreover it is NM_4=2GA', and then AA'=2NM_4=4GA' and AG=3GA'.

Therefore G lies on the segment AA' and it is such that AG=3GA'; the same holds for the segments BB', CC', DD' and it is BG=3GB', CG=3GC', DG=3GD' (in the proof you should consider the segments M_1M_3, M_2M_4 and M_1M_3 respectively). Note that the point G bisects the two segments M_1M_3, M_2M_4 and therefore is the same point G as in Problem 1. This point is knows as centroid of a quadrilateral.

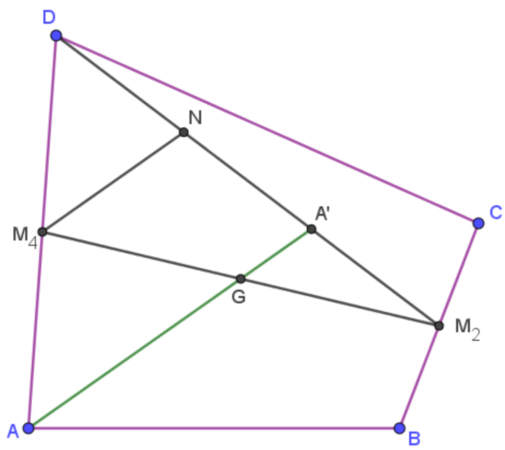

Figure 8. *Q* and the segment AA'