PROBLEM CORNER

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Problem 1

The ABCD is a cyclic quadrilateral. The line containing the segment AD and the line containing the segment BC intersect at point M. The line containing the segment AB and the line containing segment DC intersect at point P. The angles bisectors intersect the sides of the quadrilateral in the points: K, I, N, G (see Figure 1). Prove that: (1) PK= PN, (2) $\angle MOP$ = 90, (3) KING quadrilateral is a rhombus.

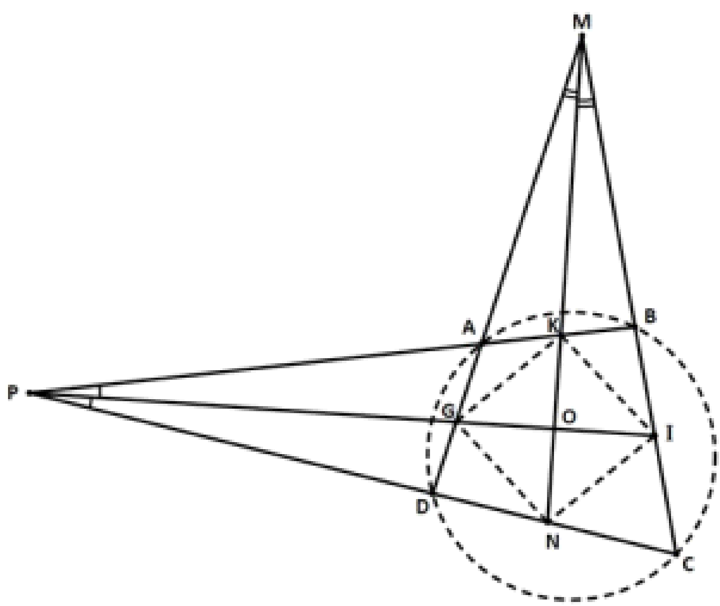


Figure 1.

Solution:

It is easy to prove (1) and (3) by using proof of (2). In order to prove (2), some angles are named by letters to make it easier to solve.

- $\angle ADC$ = Q , $\angle ABC$= H , $\angle DCB$ = L , $\angle DAB$ = J, $\angle BPC$ = 2F , $\angle DMC$ = 2E,

$\angle MNC$ = V, $\angle PIC$ = Z , $\angle MOP$ = X

- Q + H = 180° , L + J = 180° 2F + H + L = 180° 2E + Q + L = 180°

- Sum of last two given equations which is 2F + H + L + 2E + Q + L = 360°, since Q + H = 180°, we see 2F + 2L + 2E = 180° and F + L +E = 90°

- F + Z + L = 180° and V + L + E = 180°, sum of them is F + Z + L + V + L + E = 360° , Since we found that F + L +E = 90° , then Z + L + V = 270°

- Finally, X + Z + L + V = 360°, since Z + L + V = 270° , then X = 90° , it is proved that $\angle MOP$ = 90°

After the proof of (2), we can easily prove (1) by using (2):

Since $\angle MOP$ = 90°, PO is the height of KPN triangle and since $\angle OPN = \angle OPK$, it is trivial to see NO = OK, which gives PN = PK.

The solution to (3) is to make use of the proof of (2):

Since $\angle MOP$ = 90°, MO is the height of IMG triangle as well, and since $\angle MGI = \angle MIG$,

GO= OI, GK = KI and GK = GN, we see KI = GN = GK. Obviously, NI should also be equal to KI = GN = GK and since its diagonals are perpendicular as well, KING is clearly a rhombus.

Problem 2

Three circles with the same radius *R* and centers at points O_1, O_2 and O_3 are all intersected at one point *D*. Circles $O_1$and O_2 are also intersected at the point *A*. Circles O_1 and O_3 are also intersected at point *B* and circles O_2 and O_3 are also intersected at point *C*, as shown in the Figure 2. It must be proved that the circle passing through the 3 points *A*, *B*, *C* has always (constantly) the same radius *R*, when there is a change in the location of the intersection points *A* or *B* or *C*.

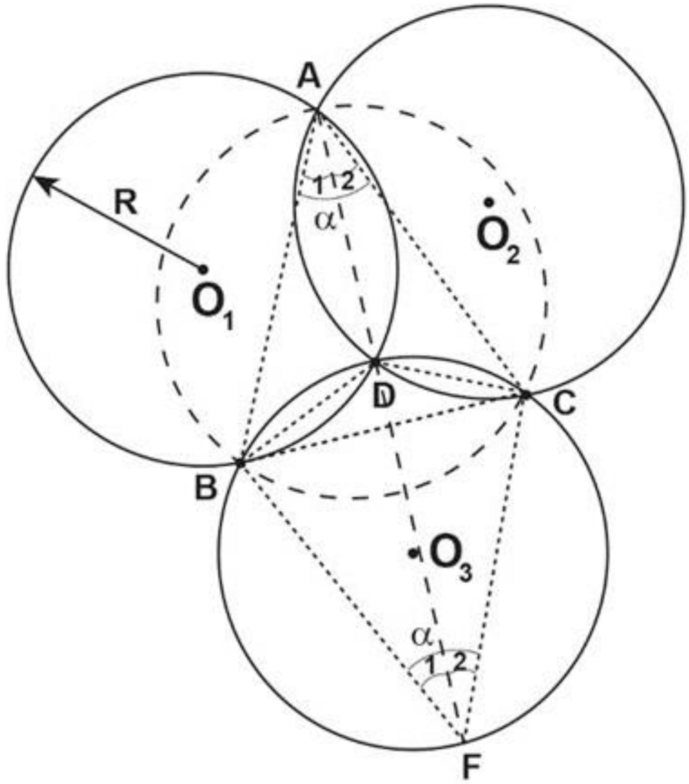
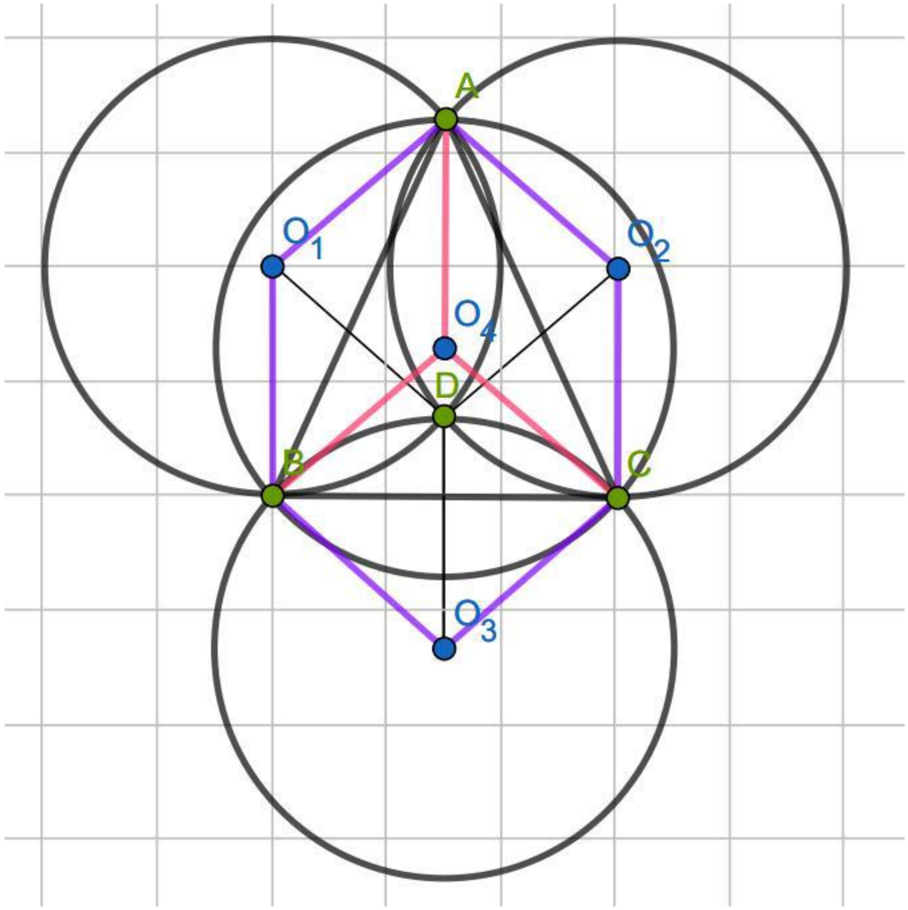


Figure 2

Solution:

Let the circles with the centers O_1, O_2, O_3 have the same radius R. We observe that O_1BO_3D is a rhombus, because $O_1B = BO_3 = O_3D = O_1D = R$. Then $O_1B \parallel O_3D$ and $O_1D \parallel BO_3$, according to the properties of the rhombus. $AO_2 \parallel O_1D$ by the same property, and we conclude that $AO_2 \parallel O_1D \parallel BO_3$ $\Rightarrow AO_2 \parallel BO_3$. By the same approach, it can be proved that $O_1A \parallel CO_3$, $BO_1 \parallel CO_2$. At the same time, we observe that $O_1A = CO_3 = BO_1 = CO_2 = AO_2 = BO_3 = R$. We can create the rhombus AO_1BO_4 from the triangle AO_1B. Then AO_4CO_2 is also a rhombus, because $AO_1 = O_1B = AO_4 = O_4B$ $= AO_2 = O_2C = R$. Thus, $AO_4 = BO_4 = CO_4 = R$, which means that O_4 is the center of the circumcircle of the triangle $ABC \Rightarrow$ the radius of this circle is also R. We have proved that the radius of the circle with the center O_4 is equal to the radii of the circles with centers O_1, O_2, O_3. This condition is true, whenever the radii of three intersecting circles are equal, and does not depend on the change of location of the intersection points.


A
O1
O2
O4
D
B
C
O3