PROBLEM CORNER

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Problem 1: Consider a parallelogram ABCD. Construct squares on all four sides and mark their centres. Join these centres to form a quadrilateral. What conjectures can you regarding this quadrilateral? Can you prove your conjecture?

(This problem may be attempted using Dynamic Geometry Software such as GeoGebra)

Solution:

Step 1: Construct a parallelogram ABCD using GeoGebra's construction tools.
(Draw a line segment AB using the **Segment** tool. Take a point C above AB. Join B to C. Through C draw a line parallel to AB using the **Parallel line** tool. Through A draw a line parallel to BC, again using the **Parallel line** tool. Use the **Intersect** tool to identify the fourth vertex D of the parallelogram. Using the **Polygon** tool construct the parallelogram ABCD. Hide all other elements of the figure using the hide/unhide option. The output is as shown in Figure 1.)

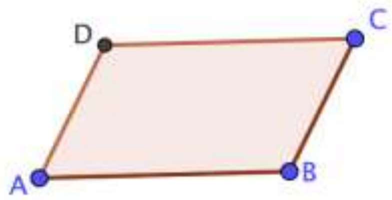

Figure 1: A parallelogram constructed using GeoGebra.

Step 2: Use the **Regular Polygon** tool to construct squares on all four sides of parallelogram ABCD.
(Select the **Regular Polygon** tool, click on the points B and C and enter the number of vertices as 4. This produces a square on side BC of the parallelogram. Repeat this process on all four sides. See Figure 2

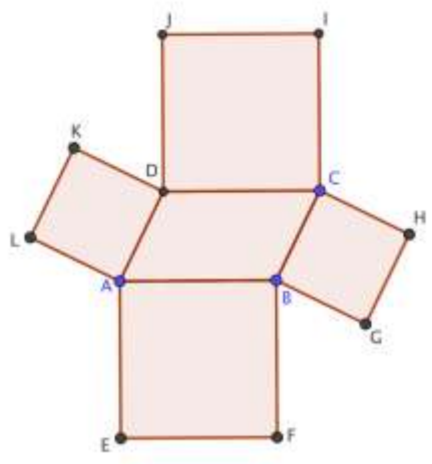

Figure 2: Squares are drawn on all four sides of the parallelogram ABCD.

Step 3: Using the **Midpoint or center** tool mark the centres of the fours squares as M, N, O and P respectively and join these using the **Segment** tool. A quadrilateral MNOP is formed. Mark the sides of MNOP using a different colour. See Figure 3.

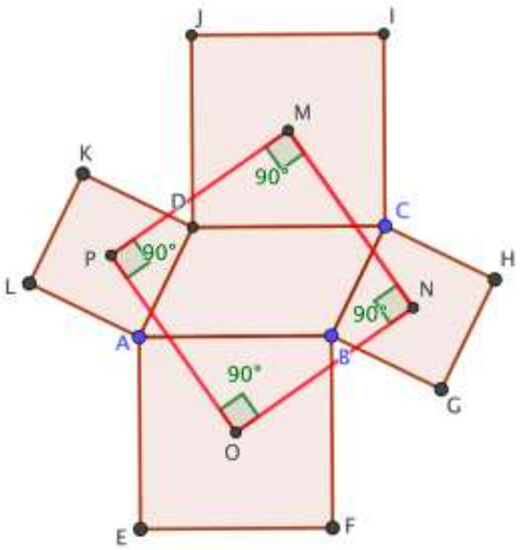

Figure 3: A quadrilateral MNOP is obtained by joining the centres of the fours squares raised on the sides of a parallelogram ABCD.

Step 4: Drag the vertices of the parallelogram ABCD and observe what happens to the quadrilateral MNOP. Upon dragging, the quadrilateral MNOP appears to be a square. To verify this conjecture, use the **Angle tool** to measure the four angles. These turn out to be right angles. Further it may be checked from the **Algebra** view that all four sides are equal. Also, by dragging the vertices A,B,C or D, these properties OF MNOP (that is, equality of sides and that all angles are right angles) remain invariant, although the side lengths of MNOP vary. Figure 4 shows another illustration where MNOP is still a square.

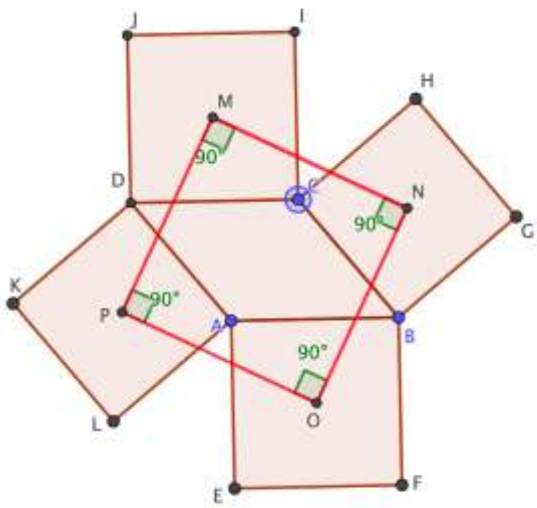

Figure 4: MNOP remains a square irrespective of the parallelogram ABCD.

Step 5: Using maintaining dragging strategies one can create special types of parallelograms ABCD. Let us consider the special case when ABCD is a rectangle. Note the NMOP is still a square. As the vertices of ABCD are dragged, MNOP preserves the properties of a square. This leads us to make the conjecture with confidence that MNOP is a square, irrespective of the type of parallelogram ABCD.

So far we have only arrived at a conjecture through empirical verification.

<u>The Proof</u>

Consider the *special* case when ABCD is a rectangle.

Note that for a given square (on one of the sides of the rectangle ABCD), the line joining the centre to two of its vertices are perpendicular and equal in length. For example, in the square with centre M, MC and MD are equal as these are half the length of the diagonals of the square. Further, angle CMD is a right angle. By extending this argument to all four squares we can conclude that the angles BNC, AOB and APD are right angles. Also, MC = MD, NB = NC, OA = OB and PA = PD. Now consider the sides MN and ON of the quadrilateral MNOP. MC = OB and NC = NB. Thus NC + CN = NB + BO. This leads to the equality of the sides NM and ON. Using similar arguments, it can be shown that all the four sides of MNOP are equal, leading to the fact that MNOP is a square. See Figure 5

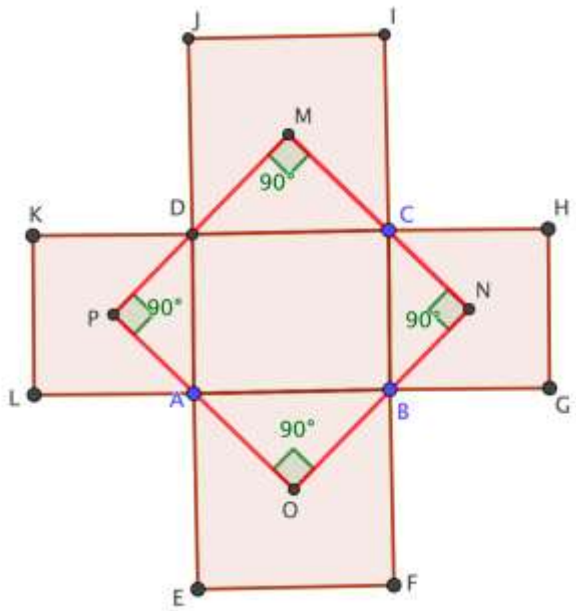

Figure 5: Special case of the problem when ABCD is a rectangle.

Let us now consider the *general* case when ABCD is any parallelogram.

Join DP, DM, CN and CM (indicated by dashed lines in Figure 6). Consider the triangles DPM and CNM. If these triangles are proven to be congruent then the equality of the adjacent sides MP and MN of quadrilateral MNOP can be concluded. Further this argument may be extended to all pairs of adjacent sides of MNOP.

In triangles DPM and CNM, MC = MD and DP = CN. Further, angle MDP = angle MCN. To prove this let us resort to algebra. Let angle BCD be x°. Then angle ADC is (180 – x)°. Now angle MCN = 45° + 45°+ x° = (90 + x)°. Using this we can conclude that angle MDP = 360° - (45° + 45°+ 180 – x°) = (90 + x)°.

Hence the triangles DPM and CNM are congruent and MP = MN. Similar triangles can be drawn on other pairs of adjacent sides leading to the conclusion that all four sides of MNOP are equal. What needs to be proved now is that all its angles are right angles.

Assume angle CMN of triangle CMN to be y°. Thus the measure of angle DMN is (90 – y)°. However, due to congruency of the triangles DPM and CNM, angles DMP and CMN are equal. This leads to the fact that angle NMP is (90 – y)° + y°, that is, 90°. This argument may be extended to show that all four angles of MNOP are 90°. Thus MNOP is a square. Dragging the vertices of the parallelogram ABCD preserves the properties of MNOP proving that it is a square.

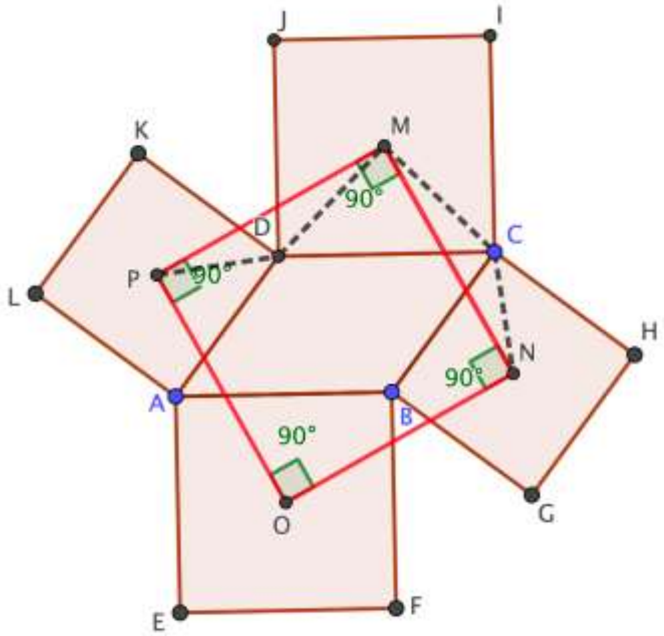

Figure 6: Congruency of the triangles DMP and CNM help to prove that MNOP is a square.

Problem 2: We are all aware of the polygonal number series. Among them the triangular, square and pentagonal number series are as follows:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55,…

1, 4, 9, 16, 25, 36, 49, 64, 81, 100,…

1, 5, 12, 22, 35, 51, 70, 92, 107, 135,…

There are numbers which belong to two of the above series. For example, 36 is both a triangular number as well as a square number. Such numbers are called *triangular square numbers*. Similarly, there are numbers which are *pentagonal square numbers*.

(i) Generate the series of the first 1000 triangular, square and pentagonal numbers. (You may use a spreadsheet such as MS Excel.)

(ii) Identify the triangular square numbers and the pentagonal square numbers from the date generated in (i).

(iii) Is it possible to find a rule for generating both types of numbers without using a spreadsheet?

Solution:

(i) The nth term of the triangular number series is $\frac{n(n+1)}{2}$

The nth term of the square number series is n^2

The nth term of the pentagonal number series is $\frac{3n^2-n}{2}$

The following steps may be used to generate the three series of numbers in Excel.

Step 1: To generate these series, create a sequence of numbers from 1 to 1000 in column A. (Enter 1 in cell A2, followed by =A2+1 in cell A3. Using the **Fill Series** option we can create a column of numbers from 1 to 1000).

Step 2: Enter the formulae =A2*(A2+1)/2, =A2*A2 and =(3*A2^2-A2)/2 in cells B2, C2 and D2 respectively. Selecting these cells simultaneously, followed by double clicking, leads to the generation of the first 1000 triangular, square and pentagonal numbers in columns B, C and D respectively. See Figure 7 which shows the first 20 and the last 20 numbers of all three series on an Excel sheet.

	A	B	C	D
1	S.No.	Triangular nos.	Square nos.	Pentagonal nos.
2	1	1	1	1
3	2	3	4	5
4	3	6	9	12
5	4	10	16	22
6	5	15	25	35
7	6	21	36	51
8	7	28	49	70
9	8	36	64	92
10	9	45	81	117
11	10	55	100	145
12	11	66	121	176
13	12	78	144	210
14	13	91	169	247
15	14	105	196	287
16	15	120	225	330
17	16	136	256	376
18	17	153	289	425
19	18	171	324	477
20	19	190	361	532
21	20	210	400	590

	A	B	C	D
982	981	481671	962361	1443051
983	982	482653	964324	1445995
984	983	483636	966289	1448942
985	984	484620	968256	1451892
986	985	485605	970225	1454845
987	986	486591	972196	1457801
988	987	487578	974169	1460760
989	988	488566	976144	1463722
990	989	489555	978121	1466687
991	990	490545	980100	1469655
992	991	491536	982081	1472626
993	992	492528	984064	1475600
994	993	493521	986049	1478577
995	994	494515	988036	1481557
996	995	495510	990025	1484540
997	996	496506	992016	1487526
998	997	497503	994009	1490515
999	998	498501	996004	1493507
000	999	499500	998001	1496502
001	1000	500500	1000000	1499500

Figure 7: The first 20 and the last 20 numbers of the triangular, square and pentagonal number series (upto their 1000^{th} terms)

(iii)To identify the common values among the triangular and square numbers, we use the **Conditional Formatting** option in Excel. To use this feature, select columns B and C, then go to the **Conditional Formatting** option, select **Highlight cell rules** and then **Duplicate values**.

Note that Excel highlights the common numbers in these two series. Thus, the triangular square numbers identified are 36, 1225 and 41616.

36 is the 6^{th} square number and the 8^{th} triangular number. 1225 is the 35^{th} square number and the 49^{th} triangular number. 41616 is the 204^{th} square number and the 288^{th} triangular number. Figure 8 shows the Excel screenshots.

A	B	C
S.No.	Triangular nos.	Square nos.
1	1	1
2	3	4
3	6	9
4	10	16
5	15	25
6	21	36
7	28	49
8	36	64
9	45	81
10	55	100

34	595	1156
35	630	1225
36	666	1296
37	703	1369
38	741	1444
39	780	1521
40	820	1600
41	861	1681
42	903	1764
43	946	1849
44	990	1936
45	1035	2025
46	1081	2116
47	1128	2209
48	1176	2304
49	1225	2401

203	20706	41209
204	20910	41616
205	21115	42025
206	21321	42436
207	21528	42849
208	21736	43264
209	21945	43681
210	22155	44100

286	41041	81796
287	41328	82369
288	41616	82944
289	41905	83521
290	42195	84100
291	42486	84681
292	42778	85264
293	43071	85849
294	43365	86436
295	43660	87025

Figure 8: The first three triangular square numbers identified by Excel through **Conditional Formatting (Duplicate values)** option.

Similarly **Conditional Formatting** may be used to identify the common values among the square and pentagonal numbers. We find only one pentagonal square number! 9801 is the 81^{st} pentagonal number and the 99^{th} square number. Figure 9 shows the Excel screenshots.

80	6400	9560
81	6561	9801
82	6724	10045
83	6889	10292
84	7056	10542
85	7225	10795
86	7396	11051
87	7569	11310

95	9025	13490
96	9216	13776
97	9409	14065
98	9604	14357
99	9801	14652
100	10000	14950

Figure 9: **Conditional Formatting** is used to identify the pentagonal square number.

(ii) In order to find triangular square numbers we need to solve the equation

$$m^2 = \frac{n(n+1)}{2} \qquad (1)$$

where the left hand side of (1) represents the mth square number and the right hand side the nth triangular number. With some algebraic manipulation (1) is rewritten as

$2m^2 = n^2 + n$

Multiplying both sides by 4 we get

$8m^2 = 4n^2 + 4n$ which may be further rewritten as

$8m^2 = 4n^2 + 4n + 1 - 1 = (2n + 1)^2 - 1$

We may write $8m^2 = (2n + 1)^2 - 1$ as $(2n + 1)^2 - 2(2m)^2 = 1$

or as $x^2 - 2y^2 = 1$, where $x = 2n + 1$ and $y = 2m$ (2)

(2) is a Pell's equation, that is, an equation of the form $x^2 - dy^2 = 1$ where 'd' is a given positive non-square integer whose solutions are integer values of x and y. The solution (x, y) would lead to the values (m, n). Substituting these in (1) leads to the triangular square numbers. The following theorem (stated here without proof) allows us to solve the Pell's equation (2) using continued fractions.

Theorem: If (x,y) is a solution of $x^2 - dy^2 = 1$, then x/y is a convergent of $\sqrt{d}$.

Since $d = 2$ in the case of triangular square numbers, the convergents of $\sqrt{2}$ would lead to the required solution. A continued fraction is an expression obtained through an iterative process. It comprises a number plus a fraction whose denominator is a number plus a fraction and that fraction's denominator is also a number plus a fraction, ad infinitum.

For example, the continued fraction for $\sqrt{2}$ is $1 + \frac{1}{2+\frac{1}{2+\frac{1}{2\ldots}}}$. This may be written as

[1; 2, 2, 2,….] using the standard notation for continued fractions.

The convergents of this continued fraction are

$$1,\ 1+\frac{1}{2},\ 1+\frac{1}{2+\frac{1}{2}},\ 1+\frac{1}{2+\frac{1}{2+\frac{1}{2}}},\ldots\ldots\ldots$$

which leads to the sequence of fractions 1, $\frac{3}{2}$, $\frac{7}{5}$, $\frac{17}{12}$,… This sequence of fractions converges to $\sqrt{2}$.

For example, the solution 3/2, that is, $x = 3$ and $y = 2$, leads to $m = n = 1$ (the trivial solutions). Also 17/12, that is, $x = 17$ and $y = 12$, leads to $m = 6$ and $n = 8$. Substituting these in (1) we get the triangular square number 36. The reader may explore which convergents of $\sqrt{2}$ would lead to the triangular square numbers 1225 and 41616 which we have identified earlier using Excel.

Similarly, In order to find pentagonal square numbers we need to solve the equation

$$m^2 = \frac{3n^2 - n}{2} \qquad (3)$$

$$2m^2 = 3n^2 - n = 3\left(n^2 - \frac{n}{3}\right)$$

By completing the square we obtain

$2m^2 = 3\left(n^2 - \frac{n}{3} + \frac{1}{36}\right) - \frac{1}{12}$, which lead to

$2m^2 = 3\left(n - \frac{1}{6}\right)^2 - \frac{1}{12}$ or

$$2m^2 = 3\left(\frac{6n-1}{6}\right)^2 - \frac{1}{12}$$

Simplifying further, we get

$24m^2 = (6n - 1)^2 - 1$ or

$x^2 - 6y^2 = 1$, where $x = 6n - 1$ and $y = 2m$ (4)

According to the theorem above, we need to look for the convergents of $\sqrt{6}$, to find the pentagonal square number.

The continued fraction for $\sqrt{6}$ is $2 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2\ldots}}}$ or [2; 2, 4, 2, 4, …] in standard notation.

Its convergents are 2, $2 + \frac{1}{2}$, $2 + \cfrac{1}{2 + \frac{1}{4}}$, $2 + \cfrac{1}{2 + \cfrac{1}{4 + \frac{1}{2\ldots}}}$,…

$2, \frac{5}{2}, \frac{22}{9}, \frac{49}{20}, \frac{218}{89}, \frac{485}{198}, \ldots$

Note that the fraction $\frac{485}{198}$, that is $x = 485$ and $y = 198$ leads to $n = 81$ and $m = 99$. Substituting these values of m and n in Pell's equation (3) lead to the pentagonal square number 9801!