PROBLEM CORNER

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Background

We are given two disconnected circles, C_1, and C_2 centered at O_1 and O_2, respectively. Point $A = (x, y) \in \mathbb{R}^2$ is outside both circles, and we construct two pairs of tangent lines from A toward circles C_1 and C_2, respectively. We label those points of tangency as B, C, D, and E, respectively; (see Figure 1). If $\measuredangle BAC = \measuredangle DAE$, we call point A an equal viewpoint.

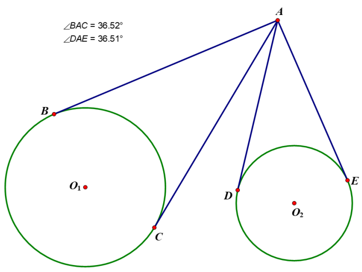

Figure 1. Equal viewpoint A.

Introduction:

Question: Find the locus of equal viewpoint A for given two circles.

We note that **Question** can be described as follows: In view of Figure 2 below,

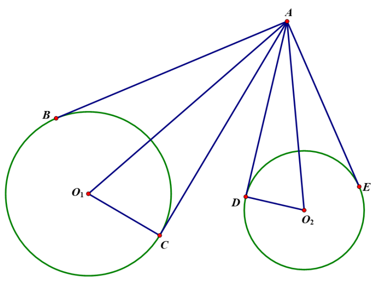

Figure 2. Using ratios to approach equal viewpoint

it follows from the assumptions that $\measuredangle O_1AC = \frac{1}{2}\measuredangle BAC = \frac{1}{2}\measuredangle DAE = \measuredangle O_2AD$, and $\measuredangle O_1CA = \measuredangle O_2DA$. Thus, $\triangle O_1AC$ is similar to $\triangle O_2AD$. Hence,

$$\frac{AO_1}{AO_2} = \frac{CO_1}{DO_2} = \frac{R_1}{R_2}, \tag{1}$$

where R_1 and R_2 are the radii of the circles C_1 and C_2 respectively.

The preceding **Question** is similar to the so-called Circle of Apollonius (See [1]).

Find the locus of those points of equal viewpoint where the ratio of the distances to the centers of two respective circles is a constant (i.e. is a constant, see Figure 2).

We label the moving locus for the equal viewpoint $A = (x, y)$, and label the centers of two non-degenerate circles $O_1(x_1, y_1)$ and $O_2(x_2, y_2)$ respectively. Furthermore, because of Eq. (1), if we assume $\frac{R_1}{R_2} = m > 0$, then $\frac{AO_1}{AO_2} = m$. Consequently, we have

$$(x - x_1)^2 + (y - y_1)^2 = m^2\left((x - x_2)^2 + (y - y_2)^2\right)$$

After reordering, we have

$$\begin{aligned}&\left(1 - m^2\right)x^2 + \left(1 - m^2\right)y^2 + 2\left(m^2x_2 - x_1\right)x + 2\left(m^2y_2 - y_1\right)y \\ &= m^2\left(x_2^2 + y_2^2\right) - \left(x_1^2 + y_1^2\right).\end{aligned} \tag{2}$$

1. Case 1. If $m \neq 1$, then

   $$x^2 + y^2 + \frac{2\left(m^2x_2 - x_1\right)}{1 - m^2}x + \frac{2\left(m^2y_2 - y_1\right)}{1 - m^2}y = \frac{m^2\left(x_2^2 + y_2^2\right) - \left(x_1^2 + y_1^2\right)}{1 - m^2}$$

   By completing square, we obtain

   $$\left(x + \frac{m^2x_2 - x_1}{1 - m^2}\right)^2 + \left(y + \frac{m^2y_2 - y_1}{1 - m^2}\right)^2 = \frac{m^2\left(x_1^2 - 2x_2x_1 + x_2^2 + y_1^2 - 2y_2y_1 + y_2^2\right)}{\left(1 - m^2\right)^2}. \tag{3}$$

   We see that Eq. (3) represents a circle with center $\left(\frac{m^2x_2 - x_1}{m^2 - 1}, \frac{m^2y_2 - y_1}{m^2 - 1}\right)$, and radius $\left|\frac{m}{m^2 - 1}\right|\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ provided that $x_2^2 - x_2x_1 + y_2^2 - y_2y_1 > 0$.

2. Case 2. If $m = 1$, then the locus, according to Eq. (2), becomes a line of

   $$2\left(x_2 - x_1\right)x + 2\left(y_2 - y_1\right)y - \left(x_2^2 + y_2^2\right) + \left(x_1^2 + y_1^2\right) = 0. \tag{4}$$

Problem 1.

We consider two circles C_1 of $x^2 + y^2 - 1 = 0$, and C_2 of $(x - 2)^2 + y^2 - 1 = 0$. Find the locus of equal viewpoints.

Solution: We observe that the circle C_1 and C_2 are of the same radius, which is shown in red and black respectively in Figure 3. It can be verified geometrically or according the Case 2 discussed above, the equal viewpoints in this case will be a vertical line, $x = 1$,which is shown in blue in Figure 3.

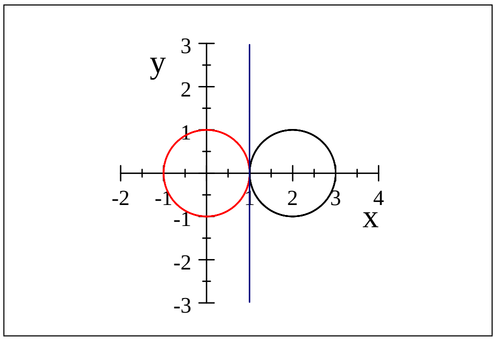

Figure 3. Equal viewpoint for two circles with the same radius

Problem 2.

We consider two non-intersecting and different sizes of circles C_1 of $x^2 + y^2 - 1 = 0$, and C_2 of $(x-4)^2 + y^2 = 4$. Find the locus of equal viewpoints.

Solution: We follow the procedure, described in the Introduction, to find that the equal viewpoint in this case is a circle C_3, of center $\left(-\frac{4}{3}, 0\right)$ and radius of $\frac{8}{3}$, which is identical to Eq. (3) where $m = \frac{1}{2}$, center is $\left(\frac{m^2x_2 - x_1}{m^2-1}, \frac{m^2y_2 - y_1}{m^2-1}\right)$, and radius $\left|\frac{m}{m^2-1}\right| \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.We depict the circle C_1, C_2 and C_3, respectively, in red, black and blue in Figure 4 below:

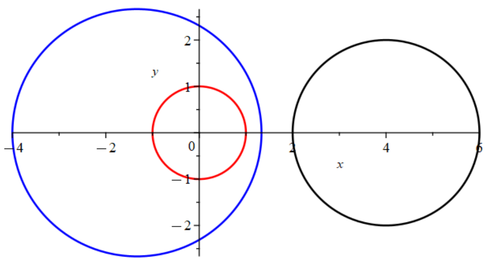

Figure 4. Equal viewpoint for two different radii

Problem 3.

We consider two intersecting circles with different radii of circles C_1 of $x^2 + y^2 - 1 = 0$, and C_2 of $(x-2)^2 + y^2 = 4$. Find the locus of equal viewpoints.

Solution: We note that the circles C_1 and C_2 are of different radii and intersecting each other. If we follow procedure described in the Introduction, the true locus should be another circle, similar to the case in **Problem 2**. However, in view of the definition of equal viewpoint, the equal viewpoint should not be contained inside the intersection between C_1 and C_2. Therefore, the locus for the equal viewpoint, in this case, is the portion shown in green in Figure 5 below.

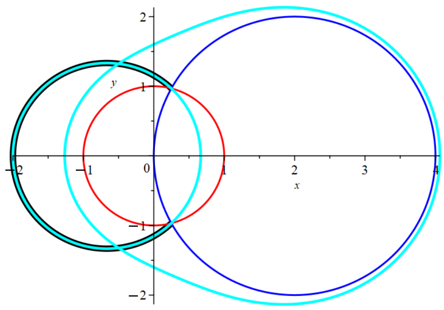

Figure 5. The locus shown in dark green

Remark: This article is extracted from the paper published in ([2]), which we discussed loci of equal viewpoints for two ellipses.

References

[1] Circle of Apollonius, see https://en.wikipedia.org/wiki/Circle#Circle_of_Apollonius/.

[2] Yang, W.-C., *Loci of Equal Viewpoints for Two Ellipses,* (https://atcm.mathandtech.org/EP2025/invited/22224.pdf). Proceedings of the 30th Asian Technology Conference in Mathematics, December 2025, ISSN 1940-4204, Mathematics and Technology, LLC.